DYNAMICS IN DUMBBELL DOMAINS III. CONTINUITY OF ATTRACTORS

JOSÉ M. ARRIETA[A,†], ALEXANDRE N. CARVALHO[B,‡], AND GERMAN LOZADA-CRUZ[C,*]

[A] Dep. Matemática Aplicada, Fac. Matemáticas, Univ. Complutense de Madrid, 28040 Madrid, Spain

[B] Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

[C] Departamento de Matemática, IBILCE, UNESP-Universidade Estadual Paulista, 15054-000 São José do Rio Preto SP, Brazil

Abstract. In this paper we conclude the analysis started in [3] and continued in [4] concerning the behavior of the asymptotic dynamics of a dissipative reactions diffusion equation in a dumbbell domain as the channel shrinks to a line segment. In [3], we have established an appropriate functional analytic framework to address this problem and we have shown the continuity of the set of equilibria. In [4], we have analyzed the behavior of the limiting problem. In this paper we show that the attractors are upper semicontinuous and, moreover, if all equilibria of the limiting problem are hyperbolic, then they are lower semicontinuous and therefore, continuous. The continuity is obtained in L^p and H^1 norms.

1. Introduction

This paper is concerned with the continuity of the asymptotic dynamics of a dissipative reaction-diffusion equation in a dumbbell type domain as the channel degenerates to a line segment. Here we conclude the analysis started in [3], where we studied the continuity of the equilibria, and continued in [4], where we studied the limiting problem. We refer to the introduction in [3] for a broad perspective of the problem.

More precisely, we consider a reaction-diffusion equation of the form

$$\begin{cases} u_t - \Delta u + u = f(u) & x \in \Omega_\varepsilon \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\varepsilon \end{cases} \tag{1.1}$$

where, for $N \geqslant 2$ and $\varepsilon \in (0,1]$, $\Omega_\varepsilon \subset \mathbb{R}^N$ is a typical dumbbell domain; that is, two disconnected domains, denoted by Ω, joined by a thin channel, denoted by R_ε. The channel R_ε degenerates to a line segment as the parameter ε approaches zero, see Figure 1. We refer to [3] Section 2, for a complete and rigorous definition of the dumbbell domain that we are

†Partially supported by Grants PHB2006-003-PC and MTM2006-08262 from MEC and by "Programa de Financiación de Grupos de Investigación UCM-Comunidad de Madrid CCG07-UCM/ESP-2393. Grupo 920894" and SIMUMAT-Comunidad de Madrid, Spain .

‡Partially supported by CNPq 305447/2005-0 and 451761/2008-1, CAPES/DGU 267/2008 and FAPESP 2008/53094-4, Brazil.

*Partially supported FAPESP # 06/04781-3 and 07/00981-0, Brazil.

considering. We mention that the channels R_ε considered here are fairly general and are not required to be cylindrical.

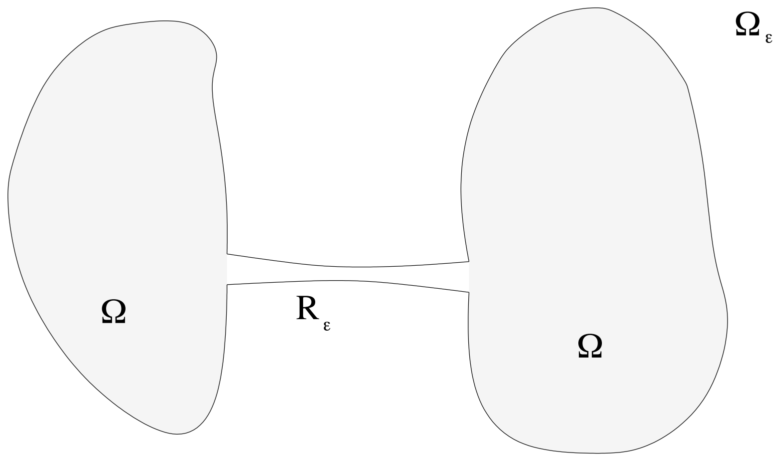

FIGURE 1. Dumbbell domain

The limit "domain" consists of the fixed part Ω and the line segment R_0. Without loss of generality, we may assume that $R_0 = \{(x, 0, \dots, 0) : 0 < x < 1\}$, see Figure 2 of [4].

The limit equation is given by

$$\begin{cases} w_t - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \\ \\ v_t - \dfrac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1) \end{cases} \tag{1.2}$$

where w is defined in Ω, v is defined in R_0 and P_0, P_1 are the points where the line segment touches the boundary of Ω. Observe that the boundary conditions of v in $(0,1)$ are given in terms of a continuity condition, so that the whole function (w, v) is continuous in the junction between Ω and R_0. The function $g : [0,1] \to (0,\infty)$ is a smooth function related to the geometry of the channel R_ε, more exactly, on the way the channel R_ε collapses to the segment line R_0, see [3]. For instance, if the channel is given by $R_\varepsilon = \{(x, \varepsilon x') : (x, x') \in R_1\}$, for some fixed reference channel R_1, then $g(x) = |\{x' : (x, x') \in R_1\}|_{N-1}$, where $|\cdot|_{N-1}$ is the $(N-1)$-dimensional Lebesgue measure, see [3].

In [3] we have studied how the equilibria of (1.1) behave as the parameter ε tends to zero. Since the spaces to which the equilibria belong also vary with ε, we developed an appropriate functional analytical setting to compare these functions as well as deal with this singular perturbation problem. We have constructed the family of spaces U^p_ε, $0 < \varepsilon \leqslant 1$, in Ω_ε, which is the space $L^p(\Omega_\varepsilon)$ with the norm

$$\|u_\varepsilon\|^p_{U^p_\varepsilon} = \int_\Omega |u|^p + \frac{1}{\varepsilon^{N-1}} \int_{R_\varepsilon} |u_\varepsilon|^p.$$

Observe that the integral in R_ε has the weight $1/\varepsilon^{N-1}$, which amplifies the effect of a function in the channel. As observed in [3] a constant function in R_ε will converge to zero if we do not introduce the appropriate weight $(1/\varepsilon^{N-1})$. In this setting, we showed that the appropriate limit space should be $U_0^p = L^p(\Omega) \oplus L_g^p(0,1)$; that is, $(w,v) \in U_0^p$ iff $w \in L^p(\Omega)$, $v \in L^p(0,1)$. The norm in U_0^p is given by

$$\|(w,v)\|_{U_0^p}^p = \int_\Omega |w|^p + \int_0^1 g|v|^p.$$

If $A_\varepsilon : D(A_\varepsilon) \subset U_\varepsilon^p \to U_\varepsilon^p$ is given by $A_\varepsilon(u) = -\Delta u + u$ for $0 < \varepsilon \leqslant 1$, and $A_0 : D(A_0) \subset U_0^p \to U_0^p$ is given by $A_0(w,v) = (-\Delta u + u, -\frac{1}{g}(gv_x)_x + v)$, we proved in Proposition 2.7 of [3] that $A_\varepsilon^{-1} \overset{\varepsilon\to 0}{\longrightarrow} A_0^{-1}$. Moreover, considering the equilibria of (1.1) and (1.2), in an abstract way, as the solutions of

$$A_\varepsilon u = F_\varepsilon(u),\ \varepsilon \in [0,1],$$

with F_ε being suitable Nemitskiĭ maps, or as fixed points of the nonlinear maps $A_\varepsilon^{-1} \circ F_\varepsilon : U_\varepsilon^p \to U_\varepsilon^p$, we showed the convergence of the equilibria see Theorem 2.3 of [3]. Also, if the equilibria of the limiting problem (1.2) are hyperbolic, we proved the convergence of the resolvent of linearizations around the equilibria and the convergence of the linear unstable manifolds.

In [4] we studied in detail the properties of the limiting problem in terms of generation of linear singular semigroups by the operator A_0, local well posedness and existence of attractor for the associated singular nonlinear semigroup. We also show that, when all equilibria are hyperbolic, the attractor of the limiting problem (which is not gradient) can be characterized as the union of the unstable manifolds of the equilibria.

As we mentioned in the introduction of [3], our final objective is to compare the whole dynamics of problems (1.1) and (1.2). That is, to prove the continuity of the attractors as ε tends to zero. To accomplish this goal, we proposed an agenda based on a deep and thorough study of the linear part of the problems consisting on the study of the convergence properties of the resolvent operators. That agenda was established in the introduction of [3] and consisted of six items. The first three were covered in [3].

In this paper we consider the last three items of that agenda and complete the analysis. Hence, we show the convergence of the resolvent operators $(\lambda + A_\varepsilon)^{-1}$ to $(\lambda + A_0)^{-1}$ and use this information to obtain the convergence of the linear semigroups. With the variation of constants formula and the convergence of linear semigroups we show the convergence of the nonlinear semigroups, from which the upper semicontinuity of the attractors follows easily. This is done in a very similar manner as in [2].

Finally, if each equilibria of the limitting problem is hyperbolic, with the convergence of the equilibria and of its linear unstable manifolds, we show the convergence of the local nonlinear unstable manifolds of equilibria. Using the gradient-like structure of the limiting equation we prove lower semicontinuity (and therefore the continuity) of the attractors.

Next, we describe contents of the paper. In Section 2 we recall the general setting of the problem and state the main results of this paper; that is, the upper and lower semicontinuity of the attractors. In Section 3 we study the convergence of the resolvent operators associated with the linear operators obtaining rates of convergence of equilibria and of resolvent operators associated to the linearizations around equilibria. Based in the resolvent estimates obtained in Section 3, we analyze in Section 4 the convergence of the linear semigroups. In Section 5 we obtain the continuity of the nonlinear semigroups and the upper semicontinuity of the attractors. In Section 6 and, under the assumption that all equilibria of the limiting problem are hyperbolic, prove that the local unstable manifolds behave continuously as ε tends to zero. The continuity of local unstable manifolds is the key step to show the continuity of the attractors. Finally in Section 7, we analyze the continuity properties of the attractors in other norms.

Acknowledgement. We thank Antonio L. Pereira for several helpful comments on the estimates of Sections 3 and 4.

Special dedication. The question of the continuity of attractors of reaction-diffusion equations in dumbbell domains, as it is addressed in this paper as well as in [3, 4], was raised by Jack K. Hale and a great amount of the ideas and techniques explored in the three articles were proposed initially by him. The authors are specially grateful for his permanent support and motivation and would like to dedicate this work to him on the occasion of his 80th birthday.

2. Setting of the problem and statement of the main results

The setting is the same as the one we established initially in [3]. We recall some of the terminology which will be needed to study the continuity of attractors.

Consider the spaces U^p_ε and U^p_0 defined in Section 1, see also [3]. Let $0 < \varepsilon \leqslant 1$ and let $A_\varepsilon : D(A_\varepsilon) \subset U^p_\varepsilon \to U^p_\varepsilon$, $1 \leqslant p < \infty$, be the linear operator defined by

$$\begin{aligned} D(A_\varepsilon) &= \left\{u \in W^{2,p}(\Omega_\varepsilon) : \Delta u \in U^p_\varepsilon,\ \partial u/\partial n = 0 \text{ in } \partial\Omega_\varepsilon\right\}, \\ A_\varepsilon u &= -\Delta u + u,\ \ u \in D(A_\varepsilon). \end{aligned} \tag{2.1}$$

Also, for $p > \frac{N}{2}$, let $A_0 : D(A_0) \subset U^p_0 \to U^p_0$ be the operator defined by

$$D(A_0) = \left\{(w,v) \in U^p_0 : w \in D(\Delta^\Omega_N),\ (gv')' \in L^p(0,1),\ v(0) = w(P_0),\ v(1) = w(P_1)\right\} \tag{2.2}$$

$$A_0(w,v) = \Big(-\Delta w + w, -\frac{1}{g}(gv')' + v\Big), \quad (w,v) \in D(A_0), \tag{2.3}$$

where Δ^Ω_N is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ with $D(\Delta^\Omega_N) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$.

We note that, for $p > \frac{N}{2}$ we have that $D(\Delta^\Omega_N)$ is continuously embedded in $C(\bar{\Omega})$. In that case, the functions in $D(\Delta^\Omega_N)$ have well defined traces at P_0 and P_1.

Recall that we have defined in [3] the operator $M_\varepsilon : U^p_\varepsilon \to U^p_0$, as follows

$$\psi_\varepsilon \to (M_\varepsilon\psi_\varepsilon)(z) = \begin{cases} \psi_\varepsilon(z), & z \in \Omega \\ \\ \dfrac{1}{|\Gamma^z_\varepsilon|}\displaystyle\int_{\Gamma^z_\varepsilon} \psi(z,y)dy, & z \in (0,1), \end{cases} \tag{2.4}$$

where $\Gamma^z_\varepsilon = \{y : (z,y) \in R_\varepsilon\}$. It is easy to see, from Fubini-Tonelli Theorem and Hölder inequality, that M_ε is a well defined bounded linear operator with $\|M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^p_0)} = 1$.

Also consider the family of extension operators $E_\varepsilon : U_0^p \to U_\varepsilon^p$ defined by

$$E_\varepsilon(w,v)(x) = \begin{cases} w(x), & x \in \Omega \\ v(s), & (s,y) \in R_\varepsilon. \end{cases} \tag{2.5}$$

It is very easy to see that $\|E_\varepsilon(w,v)\|_{U_\varepsilon^p} = \|(w,v)\|_{U_0^p}$.

The operator A_ε generates an analytic semigroup $\{e^{A_\varepsilon t} : t \geqslant 0\}$ on U_ε^p whereas, from the results in [4], the operator A_0 generates a *singular semigroup* in U_0^p that we will denote by $\{e^{-A_0 t} : t \geqslant 0\}$, see [4].

We rewrite (1.1) and (1.2) in the abstract form

$$\begin{cases} \dot{u}_\varepsilon + A_\varepsilon u_\varepsilon = f_\varepsilon(u_\varepsilon) \\ u_\varepsilon(0) = u_0^\varepsilon \in U_\varepsilon^p \end{cases} \tag{2.6}$$

and

$$\begin{cases} \dot{u} + A_0 u = f_0(u) \\ u(0) = u_0 \in U_0^p \end{cases} \tag{2.7}$$

With respect to the nonlinearity f, we will assume that

(i) $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function,
(ii) $|f(u)| + |f'(u)| + |f''(u)| \leqslant C_1$ for all $u \in \mathbb{R}$.

Remark 2.1. *From the point of view of studying the asymptotic dynamics (continuity of attractors), the assumption* (ii) *does not imply any restriction on the nonlinearities. Since we are assuming that f is dissipative, under the usual growth assumptions, the attractors are bounded in $L^\infty(\Omega_\varepsilon)$ uniformly with respect to $\varepsilon \in [0,1]$ and one may cut the nonlinearities to make them satisfy the above assumptions (See Remark 2.2 of* [3]*).*

Under these assumptions, the nonlinear semigroups $\{T_\varepsilon(t) : t \geqslant 0\}$ in U_ε^p associated with (2.6) and the singular semigroup $\{T_0(t) : t \geqslant 0\}$ in U_0^p, $p > N/2$, associated with (2.7), have compact global attractors $\mathcal{A}_\varepsilon \subset U_\varepsilon^p$ and $\mathcal{A}_0 \subset U_0^p$ respectively (see [4]). In general, the attractors lie in more regular spaces and in particular, from comparison arguments, they lie in U_ε^∞ and U_0^∞.

The following concept of E-convergence has been proved to be very appropriate when dealing with sequences of functions in different spaces, see [14, 7, 3].

Definition 2.2. *We say that a sequence $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$, $u_\varepsilon \in U_\varepsilon^p$, $E_\varepsilon-$converges to $u_0 \in U_0^p$ if $\|u_\varepsilon - E_\varepsilon u_0\|_{U_\varepsilon^p} \xrightarrow{\varepsilon \to 0} 0$ (see* (2.5) *for the definition of E_ε). We write this as $u_\varepsilon \xrightarrow{E} u_0$.*

This notion of convergence can be extended to sets in the following manner (see [7]).

Definition 2.3. *Let $\mathcal{A}_\varepsilon \subset U_\varepsilon^p$, $\varepsilon \in [0,1]$ and $\mathcal{A}_0 = \mathcal{A} \subset U_0^p$. Denote by* $\operatorname{dist}(\cdot,\cdot)$ *the metric induced by the norm in U_ε^p, $\varepsilon \in [0,1]$, i.e.* $\operatorname{dist}(u_\varepsilon, v_\varepsilon) = \|u_\varepsilon - v_\varepsilon\|_{U_\varepsilon^p}$.

1. *We say that the family of sets $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ is E_ε-upper semicontinuous at $\varepsilon = 0$ if* $\sup_{u_\varepsilon \in \mathcal{A}_\varepsilon} \operatorname{dist}(u_\varepsilon, E_\varepsilon \mathcal{A}) \xrightarrow{\varepsilon \to 0} 0$.
2. *We say that the family of sets $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ is E_ε-lower semicontinuous at $\varepsilon = 0$ if* $\sup_{u \in \mathcal{A}} \operatorname{dist}(E_\varepsilon u, \mathcal{A}_\varepsilon) \xrightarrow{\varepsilon \to 0} 0$.

Remark 2.4. *In order to show the upper or lower semicontinuity of sets, the following characterizations are usefull*

(1) *If any sequence $\{u_\varepsilon\}$ with $u_\varepsilon \in \mathcal{A}_\varepsilon$ has an $E_\varepsilon-$convergent subsequence with limit belonging to $\mathcal{A}$, then $\{\mathcal{A}_\varepsilon\}$ is $E_\varepsilon-$upper semicontinuous at zero.*
(2) *If $\mathcal{A}$ is compact and for any $u \in \mathcal{A}$ there is a sequence $\{u_\varepsilon\}$ with $u_\varepsilon \in \mathcal{A}_\varepsilon$, which $E_\varepsilon-$converges to u, then $\{\mathcal{A}_\varepsilon\}$ is $E_\varepsilon-$lower semicontinuous at zero.*

With all this concepts in mind, our main result is the following,

Theorem 2.5. *The family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ is E_ε- upper semicontinuous at $\varepsilon = 0$ in U^p_ε for every $1 \leqslant p < \infty$.*

Moreover, if every equilibria of the limit problem is hyperbolic, then the family of attractors is also E_ε- lower semicontinuous at $\varepsilon = 0$ in U^p_ε for every $1 \leqslant p < \infty$.

Remark 2.6. *Observe that once the statement of Theorem 2.5 is shown for a particular $p \geqslant 1$, then from the boundedness of the attractors in U^∞_ε and U^∞_0, it will also be proved for all $1 \leqslant p < \infty$.*

Now consider the spaces $U^{1,2}_\varepsilon = W^{1,2}(\Omega) \oplus W^{1,2}(R_\varepsilon)$ with the norm

$$\|u_\varepsilon\|^2_{U^{1,2}_\varepsilon} = \|u_\varepsilon\|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon^{N-1}}\|u_\varepsilon\|^2_{W^{1,2}(R_\varepsilon)} \tag{2.8}$$

and $U^{1,2}_0 = W^{1,2}(\Omega) \oplus W^{1,2}(0,1)$ with the norm

$$\|(w,v)\|^2_{U^{1,2}_0} = \|w\|^2_{W^{1,2}(\Omega)} + \int_0^1 g(|v_x|^2 + |v|^2).$$

Observe that the spaces $U^{1,2}_\varepsilon$ do not coincide algebraically with the spaces $W^{1,2}(\Omega_\varepsilon)$ since we are allowing the functions of $U^{1,2}_\varepsilon$ to be discontinuous at $\partial\Omega \cap \partial R_\varepsilon$.

We also prove that

Theorem 2.7. *The family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ is E_ε- upper semicontinuous at $\varepsilon = 0$ in $U^{1,2}_\varepsilon$.*

Moreover, if every equilibria of the limit problem is hyperbolic, then the family of attractors is also E_ε- lower semicontinuous at $\varepsilon = 0$ in $U^{1,2}_\varepsilon$.

3. Convergence of resolvent operators

In this section we analyze the convergence of the resolvent operators associated to the elliptic operators A_ε defined in Section 2, that is, we study the convergence of $(A_\varepsilon + \lambda)^{-1} \to (A_0 + \lambda)^{-1}$ as $\varepsilon \to 0$ with λ in some region of the complex plane.

The convergence of resolvent operators is used, in Section 4, to analyze the convergence properties of the linear semigroups $e^{-A_\varepsilon t} \to e^{-A_0 t}$ as $\varepsilon \to 0$, with the aid of the expression

$$e^{-A_\varepsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(A_\varepsilon + \lambda)^{-1} d\lambda, \quad t > 0$$

where Γ is an appropriate unbounded curve in the complex plane.

Moreover, since we need to analyze also the convergence properties of the linear semigroups associated to linearized equations around equilibria, that is $e^{-(A_\varepsilon - f'(u^*_\varepsilon))t}$ to $e^{-(A_0 - f'(u^*_0))t}$ as ε

tends to 0, where u^*_ε and u^*_0 are equilibria for (2.6), and (2.7), respectively, we will also need to study the convergence properties of the resolvent operators $(A_\varepsilon+V_\varepsilon+\lambda)^{-1} \to (A_0+V_0+\lambda)^{-1}$ as $\varepsilon \to 0$ for the potentials $V_\varepsilon(x) = -f'(u^*_\varepsilon(x))$ and $V_0(x) = -f'(u^*_0(x))$. To show this convergence we will need to obtain some rates of convergence of the equilibria u^*_ϵ to u^*_0.

We have divided the section in several subsections. In Subsection 3.1, we analyze the convergence of the resolvent operators for a fixed potential and in Subsection 3.2 we analyze the case of a potential which depends on the parameter ε. In Subsection 3.3 we obtain some rates of convergence of the equilibria and use this rates to obtain the convergence of the resolvent operators of the linearized operators around the equilibria.

3.1. **Rate of convergence of resolvent operators: The case of a fixed potential.** Consider a complex potential $V_0 = (V_\Omega, V_{R_0}) \in U^\infty_0$. Often, we write V_0 for $E_\varepsilon V_0 \in L^\infty(\Omega_\varepsilon)$. Consider also the operator in $\mathcal{L}(L^p(\Omega_\varepsilon))$ and in $\mathcal{L}(U^p_0)$ which is the multiplication by the potential V_0. We denote this operator again by V_0, that is, $V_0(u_\varepsilon) \equiv (E_\varepsilon V_0)u_\varepsilon \equiv V_0 u_\varepsilon$ and $V_0(w, v) = (V_\Omega w, V_{R_0} v)$.

Let us assume that $\operatorname{Re}\sigma(A_0+V_0) > \delta > 0$. It follows from of [3, Proposition 3.13, Corollary 3.14] that, for all suitably small ε, $\operatorname{Re}\sigma(A_\varepsilon + V_0) \geqslant \delta > 0$.

The operator $A_\varepsilon + V_0$ is sectorial and the following estimate holds

$$\|\,(\lambda + A_\varepsilon + V_0)^{-1}\,\|_{\mathcal{L}(L^p(\Omega_\varepsilon))} \leqslant \frac{C}{|\lambda| + 1}, \quad \text{for} \quad \lambda \in \Sigma_\theta, \tag{3.1}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and C is a constant that does not depend on ε, although it depends on p and blows up as $p \to \infty$. This estimate follows from the fact that the localization of the numerical range in the complex plane can be done independently of ε, see [13].

We know that, for any $0 < \varepsilon \leqslant 1$, the operator $A_\varepsilon + V_0$ is a sectorial operator in U^p_ε and the following result holds

Lemma 3.1. *For any bounded linear operator $J : L^p(\Omega_\varepsilon) \to L^p(\Omega_\varepsilon)$ we have*

$$\|J\|_{\mathcal{L}(U^p_\varepsilon)} \leqslant \|J\|_{\mathcal{L}(L^p(\Omega_\varepsilon), U^p_\varepsilon)} \leqslant \varepsilon^{\frac{-N+1}{p}} \|J\|_{\mathcal{L}(L^p(\Omega_\varepsilon))} \tag{3.2}$$

Proof: The proof of this result follows immediately from the norm estimate

$$\|\cdot\|_{U^p_\varepsilon} \leqslant \varepsilon^{\frac{-N+1}{p}} \|\cdot\|_{L^p(\Omega_\varepsilon)}. \tag{3.3}$$

which follows directly from the definition of the norm in U^p_ε. ■

In particular, from Lemma 3.1 and from estimate (3.1), we have that for all $\lambda \in \Sigma_\theta$

$$\|(\lambda + A_\varepsilon + V_0)^{-1}\|_{\mathcal{L}(U^p_\varepsilon)} \leqslant \|(\lambda + A_\varepsilon + V_0)^{-1}\|_{\mathcal{L}(L^p(\Omega_\varepsilon), U^p_\varepsilon)} \leqslant C\frac{\varepsilon^{\frac{-N+1}{p}}}{|\lambda| + 1}, \quad \text{for} \quad \lambda \in \Sigma_\theta. \tag{3.4}$$

As for the limit problem, from [4], we have the following result.

Proposition 3.2. *The operator $A_0 + V_0$ defined by (2.2) has the following properties*

i) *$D(A_0 + V_0)$ is dense in U^p_0,*
ii) *$A_0 + V_0$ is a closed operator,*
iii) *$A_0 + V_0$ has compact resolvent and*

iv) *$A_0+V_0 : D(A_0+V_0)\subset U_0^p \to U_0^p$ is such that, $\rho(A_0+V_0)\supset \Sigma_\theta$ where $\Sigma_\theta=\{\lambda\in\mathbb{C} : |\arg(\lambda)|\leqslant \pi-\theta\}$, $0<\theta<\frac{\pi}{2}$, and for $p\geqslant q>\frac{N}{2}$,*

$$\|(\lambda+A_0+V_0)^{-1}\|_{\mathcal{L}(U_0^q,U_0^p)}\leqslant \frac{C}{|\lambda|^\alpha+1}, \tag{3.5}$$

$$\|(\lambda+A_0+V_0)^{-1}\|_{\mathcal{L}(U_0^\infty)}\leqslant \frac{C}{|\lambda|+1}, \tag{3.6}$$

$$\|(\lambda+A_0+V_0)^{-1}\|_{\mathcal{L}(U_0^\infty,U_0^p)}\leqslant \frac{C}{|\lambda|+1}, \tag{3.7}$$

for each $0<\alpha<1-\frac{N}{2q}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})<1$ and $\lambda\in\Sigma_\theta$.

v) *If B_0 is the realization of A_0 in $C(\bar{\Omega})\oplus L^p(0,1)$ we have that B_0 is a sectorial operator in $C(\overline{\Omega})\oplus L^p_g(0,1)$ with compact resolvent. Therefore $-B_0$ generates an analytic semigroup e^{-B_0t} in $C(\overline{\Omega})\oplus L^p_g(0,1)$.*

The following result is crucial to the remaining results in this section and to the whole program of the paper.

Proposition 3.3. *If $p>N$ and $2\leqslant q<\infty$, there is a constant C, independent of ε, such that*

$$\|A_\varepsilon^{-1}f_\varepsilon - E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{H^1(\Omega)\oplus H^1(R_\varepsilon)}\leqslant C\varepsilon^{N/2}\,\|f_\varepsilon\|_{U_\varepsilon^p}, \tag{3.8}$$

$$\|A_\varepsilon^{-1}f_\varepsilon - E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{L^q(\Omega_\varepsilon)}\leqslant C\varepsilon^{N/q}\,\|f_\varepsilon\|_{U_\varepsilon^p}, \tag{3.9}$$

and

$$\|A_\varepsilon^{-1}f_\varepsilon - E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{U_\varepsilon^q}\leqslant C\varepsilon^{1/q}\|f_\varepsilon\|_{U_\varepsilon^p}, \tag{3.10}$$

for all $f_\varepsilon\in U_\varepsilon^p$.

Proof: The inequality (3.8) was proved in Proposition A. 8 in [3]. This estimate is the key estimate for [3] and also for the complete analysis we are performing in the dumbbell domains.

Observe that in particular, from (3.8), we obtain that

$$\|A_\varepsilon^{-1}f_\varepsilon - E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{L^2(\Omega_\varepsilon)}\leqslant C\varepsilon^{N/2}\,\|f_\varepsilon\|_{U_\varepsilon^p}. \tag{3.11}$$

From [3, Lemma A.11], for $p>N/2$ we have

$$\|A_\varepsilon^{-1}f_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\|f_\varepsilon\|_{U_\varepsilon^p}. \tag{3.12}$$

Also we know that if $p>N/2$, $\|A_0^{-1}M_\varepsilon f_\varepsilon\|_{L^\infty(\Omega)\oplus L^\infty(0,1)}\leqslant C\|M_\varepsilon f_\varepsilon\|_{L^p(\Omega)\oplus L^p(0,1)}$ then

$$\|E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\|f_\varepsilon\|_{U_\varepsilon^p}. \tag{3.13}$$

which implies that

$$\|A_\varepsilon^{-1}f_\varepsilon - E_\varepsilon A_0^{-1}M_\varepsilon f_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\,\|f_\varepsilon\|_{U_\varepsilon^p}. \tag{3.14}$$

For $q\geqslant 2$, (3.9) follows from (3.11) and (3.14) and interpolation. The estimate (3.10) follows from (3.9) and (3.3). ∎

To obtain the resolvent convergence of $A_\varepsilon+V_0$ we strongly use the previous result and the following uniform (with respect to ε) estimate.

Lemma 3.4. *If V_0 is such that (A_0+V_0) is invertible, for $p>\frac{N}{2}$, we have*

$$\|E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon)}\leqslant C \tag{3.15}$$

and, for each $p>\frac{N}{2}$, there is a constant C, independent of ε, such that

$$\|E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\|_{\mathcal{L}(L^p(\Omega_\varepsilon))}\leqslant C. \tag{3.16}$$

Proof: Statement (3.15) follows from $\|E_\varepsilon\|_{\mathcal{L}(U^p_0,U^p_\varepsilon)}=\|M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon,U^p_0)}=1$ (see [3]) and from Proposition 3.2.

For (3.16) we proceed as follows. Let $f_\varepsilon\in L^p(\Omega_\varepsilon)$ and $u_\varepsilon=(w_\varepsilon,v_\varepsilon)=(A_0+V_0)^{-1}M_\varepsilon f_\varepsilon$, then

$$\begin{cases}-\Delta w_\varepsilon+w_\varepsilon+V_\Omega(x)w_\varepsilon=f_\varepsilon, & \Omega,\\ \dfrac{\partial w_\varepsilon}{\partial n}=0, & \partial\Omega\\ -\dfrac{1}{g}\left(g(v_\varepsilon)_s\right)_s+v_\varepsilon+V_{R_0}(s)v_\varepsilon=M_\varepsilon f_\varepsilon, & (0,1)\\ v_\varepsilon(0)=w_\varepsilon(P_0), \quad v_\varepsilon(1)=w_\varepsilon(P_1). \end{cases}$$

Since $p>\frac{N}{2}$, we have that

$$\|w_\varepsilon\|_{L^p(\Omega)}\leqslant C\,\|f_\varepsilon\|_{L^p(\Omega)}\quad\text{and } \|w_\varepsilon\|_{C(\bar\Omega)}\leqslant C\,\|f_\varepsilon\|_{L^p(\Omega)}.$$

In particular $|w_\varepsilon(P_0)|+|w_\varepsilon(P_1)|\leqslant C\,\|f_\varepsilon\|_{L^p(\Omega)}$. Also

$$\|v_\varepsilon\|_{L^p(0,1)}\leqslant |w_\varepsilon(P_0)|+|w_\varepsilon(P_1)|+\|M_\varepsilon f_\varepsilon\|_{L^p(0,1)}$$

and

$$\begin{aligned}\|E_\varepsilon v_\varepsilon\|_{L^p(R_\varepsilon)}&=\varepsilon^{\frac{N-1}{p}}\|v_\varepsilon\|_{L^p(0,1)}\leqslant \varepsilon^{\frac{N-1}{p}}(|w_\varepsilon(P_0)|+|w_\varepsilon(P_1)|)+\varepsilon^{\frac{N-1}{p}}\|M_\varepsilon f_\varepsilon\|_{L^p(0,1)}\\ &\leqslant |w_\varepsilon(P_0)|+|w_\varepsilon(P_1)|+\|f_\varepsilon\|_{L^p(R_\varepsilon)}\\ &\leqslant C\,\|f_\varepsilon\|_{L^p(\Omega_\varepsilon)}.\end{aligned}$$

where we have used that $\|M_\varepsilon f_\varepsilon\|_{L^p(0,1)}\leqslant \varepsilon^{-\frac{N-1}{p}}\,\|f_\varepsilon\|_{L^p(R_\varepsilon)}$. The proof is now complete. ∎

The next two lemmas are resolvent identities which allow us (together with the previous lemma) to transfer information from the resolvent convergence of A_ε to the resolvent convergence of $A_\varepsilon+V_0$.

Lemma 3.5. *If (A_0+V_0) and $(A_\varepsilon+V_0)$ are both invertible the following identity holds*

$$\begin{aligned}&(A_\varepsilon+V_0)^{-1}-E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\\ &=[I-(A_\varepsilon+V_0)^{-1}V_0](A_\varepsilon^{-1}-E_\varepsilon A_0^{-1}M_\varepsilon)[I-E_\varepsilon V_0(A_0+V_0)^{-1}M_\varepsilon].\end{aligned} \tag{3.17}$$

Proof: Since $(I-(A_\varepsilon+V_0)^{-1}V_0)(I+A_\varepsilon^{-1}V_0)=I$, the identity (3.17) is equivalent to

$$\begin{aligned}&(A_\varepsilon^{-1}-E_\varepsilon A_0^{-1}M_\varepsilon)(I-E_\varepsilon V_0(A_0+V_0)^{-1}M_\varepsilon)=\\ &\quad=(I+A_\varepsilon^{-1}V_0)((A_\varepsilon+V_0)^{-1}-E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon).\end{aligned} \tag{3.18}$$

Using that $V_0(A_0+V_0)^{-1} = I - A_0(A_0+V_0)^{-1}$ and expanding the left hand side of (3.18) we have

$$\begin{aligned}
&(A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon)(I - E_\varepsilon V_0 (A_0+V_0)^{-1} M_\varepsilon) = A_\varepsilon^{-1} - A_\varepsilon^{-1} E_\varepsilon V_0 (A_0+V_0)^{-1} M_\varepsilon \\
&- E_\varepsilon A_0^{-1} M_\varepsilon + E_\varepsilon A_0^{-1}(I - A_0(A_0+V_0)^{-1}) M_\varepsilon \\
&= A_\varepsilon^{-1} - A_\varepsilon^{-1} E_\varepsilon V_0 (A_0+V_0)^{-1} M_\varepsilon - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon
\end{aligned}$$

On the other hand, using that $A_\varepsilon^{-1} = (I + A_\varepsilon^{-1} V_0)(A_\varepsilon + V_0)^{-1}$ and expanding the right hand side of (3.18), we have

$$\begin{aligned}
&(I + A_\varepsilon^{-1} V_0)((A_\varepsilon+V_0)^{-1} - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon) \\
&= A_\varepsilon^{-1} - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon - A_\varepsilon^{-1} E_\varepsilon V_0 (A_0+V_0)^{-1} M_\varepsilon .
\end{aligned}$$

which proves (3.18). ∎

In a very similar way we also have,

Lemma 3.6. *If (A_0+V_0) and $(A_\varepsilon+V_0)$ are both invertible, the following identity holds*

$$\begin{aligned}
&(A_\varepsilon+V_0)^{-1} - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon = \\
&\quad [I - E_\varepsilon (A_0+V_0)^{-1} V_0 M_\varepsilon](A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon) \quad [I - V_0 (A_\varepsilon+V_0)^{-1}].
\end{aligned} \tag{3.19}$$

Proof: The proof is similar to the one provided for the previous lemma. ∎

We are now ready to prove the main results of this section

Proposition 3.7. *If $p, q > N$, $(A_0+V_0) : D(A_0) \subset U_0^p \to U_0^p$ has bounded inverse and $f_\varepsilon \in U_\varepsilon^p$, then*

$$\|(A_\varepsilon+V_0)^{-1} f_\varepsilon - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon f_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leqslant C\, \varepsilon^{N/q} \|f_\varepsilon\|_{U_\varepsilon^p}, \tag{3.20}$$

where C depends on $\|(A_0+V_0)^{-1}\|_{\mathcal{L}(U_0^p, U_0^p)}$ and on $\|V_0\|_{L^\infty}$, but not on ε or f_ε.

Proof: Let us start pointing out that if (A_0+V_0) is invertible, from [3] we also have that $(A_\varepsilon+V_0)$ is invertible for all suitably small ε. Hence (3.20) makes sense.

Adding and subtracting the appropriate term in (3.17) we have:

$$\begin{aligned}
&(A_\varepsilon+V_0)^{-1} - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon \\
&= (-(A_\varepsilon+V_0)^{-1} + E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon) V_0 (A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon)(I - V_0 E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon) \\
&+ (I - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon V_0)(A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon)(I - V_0 E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon).
\end{aligned}$$

Let us first estimate

$$\Theta_\varepsilon = ((A_\varepsilon+V_0)^{-1} - E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon) V_0 (A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon)(I - V_0 E_\varepsilon (A_0+V_0)^{-1} M_\varepsilon)$$

Note that, from inequality (3.10) and (3.9) we have that

$$\|A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p, U_\varepsilon^p)} \leqslant C\varepsilon^{1/p} \text{ and } \|A_\varepsilon^{-1} - E_\varepsilon A_0^{-1} M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p, L^q(\Omega_\varepsilon))} \leqslant C\varepsilon^{N/q}.$$

Since

$$\|V_0\|_{\mathcal{L}(L^q(\Omega_\varepsilon))} \leqslant C \|V_0\|_{L^\infty(\Omega_\varepsilon)} \text{ and } \|V_0\|_{\mathcal{L}(U_\varepsilon^p)} \leqslant C \|V_0\|_{L^\infty(\Omega_\varepsilon)},$$

it follows from (3.15) that

$$\|\Theta_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon)} \leqslant C\varepsilon^{1/p}\|(A_\varepsilon+V_0)^{-1} - E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))}.$$

where $C = C(\|V_0\|_{L^\infty(\Omega_\varepsilon)})$ is independent of ε. Choosing ε_0 such that $C\varepsilon^{1/p} \leqslant \frac{1}{2}$, for all $\varepsilon \in [0, \varepsilon_0]$, we have that

$$\begin{aligned}&\|(A_\varepsilon+V_0)^{-1} - E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon)}\\ &\leqslant 2\|(I - E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon V_0)(A_\varepsilon^{-1} - E_\varepsilon A_0^{-1}M_\varepsilon)(I - V_0E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon)\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))}.\end{aligned}$$

Now, from (3.15) and (3.16) there is a constant C, independent of ε, such that

$$\begin{aligned}&\|(I - V_0E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon)\|_{\mathcal{L}(U^p_\varepsilon)} \leqslant 1 + C\|V_0\|_{L^\infty(\Omega_\varepsilon)},\\ &\|(I - E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon V_0)\|_{\mathcal{L}(L^q(\Omega_\varepsilon))} \leqslant 1 + C\|V_0\|_{L^\infty(\Omega_\varepsilon)}\end{aligned}$$

Therefore, using (3.9),

$$\|(A_\varepsilon+V_0)^{-1} - E_\varepsilon(A_0+V_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \leqslant C\,\varepsilon^{N/q},$$

where the constant C does depends on $\|V_0\|_{L^\infty(\Omega_\varepsilon)}$. This shows the proposition. ■

3.2. **Rate of convergence of resolvent operators: The case of a varying potential.** We are going to study now the convergence properties of resolvent operators of the form $(A_\varepsilon + W_\varepsilon)^{-1}$ to $(A_0+W_0)^{-1}$, where W_ε converges to W_0 in a sense to be specified. We need to perform this study since we want to compare the resolvent operators of the linearizations around equilibria. Hence, we will have a family of equilibria u^*_ε which will converge to an equilibria of the limiting problem u^*_0 and we will need to consider the operators $A_\varepsilon - f'(u^*_\varepsilon)$ and $A_0 - f'(u^*_0)$ and analyze the convergence properties of their resolvent.

Having this in mind, let us consider the following setting for the potentials,

(H) $V_\varepsilon \in L^\infty(\Omega_\varepsilon)$, $V_0 = (V_\Omega, V_{R_0}) \in U^\infty_0$ be two potentials which satisfy that $|V_\varepsilon|, |V_0| \leqslant a$ for some $a > 0$ and such that for $N < q < \infty$ we have

$$\varepsilon^{\frac{-N+1}{q}}\|V_\varepsilon - E_\varepsilon V_0\|_{L^q(\Omega_\varepsilon)} \to 0, \text{ as } \varepsilon \to 0 \tag{3.21}$$

Denote by $W_\varepsilon = V_\varepsilon + a$, $W_0 = V_0 + a = (V_\Omega + a, V_{R_0} + a)$ so that W_ε and W_0 are positive and they also satisfy an estimate like (3.21) substituting V_ε and V_0 by W_ε and W_0 respectively.

As we did in Subsection 3.1, let us identify the potentials W_ε, W_0 with their corresponding multiplication operators.

With this notation and writing $\Lambda_\varepsilon = A_\varepsilon + W_\varepsilon$, we have that the operator Λ_ε is sectorial and the following estimate holds

$$\|(\lambda + \Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(L^p(\Omega_\varepsilon))} \leqslant \frac{C}{|\lambda|+1}, \quad \text{for} \quad \lambda \in \Sigma_\theta, \tag{3.22}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and C is a constant that does not depend on ε (that follows form the fact that the localization of the numerical range in the complex plane can be done independently of ε), however it depends on p and blows up as $p \to \infty$, see [13].

We know that, for any $0 < \varepsilon \leqslant 1$, the operator Λ_ε is a sectorial operator in U^p_ε and the following result holds

Lemma 3.8. *For all $\lambda \in \Sigma_\theta$ we have that*

$$\|(\lambda + \Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(U^p_\varepsilon)} \leqslant \|(\lambda + \Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(L^p(\Omega_\varepsilon), U^p_\varepsilon)} \leqslant C \frac{\varepsilon^{\frac{-N+1}{p}}}{|\lambda| + 1}. \tag{3.23}$$

Proof: It follows immediately from (3.22) and from Lemma 3.1. ■

The following result follows easily from the properties of resolvent operators. It is crucial to obtain convergence properties for resolvent operators from the convergence properties of Λ_ε^{-1} to Λ_0^{-1}.

Lemma 3.9. *As an immediate consequence of* (3.5), (3.6) *and* (3.7), *there is a constant C such that, for all $\lambda \in \Sigma_\theta$, $p \geqslant q > \frac{N}{2}$ and $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$*

$$\|E_\varepsilon(\lambda + \Lambda_0)^{-1} M_\varepsilon\|_{\mathcal{L}(U^q_\varepsilon, U^p_\varepsilon)} \leqslant \frac{C}{|\lambda|^\alpha + 1}, \tag{3.24}$$

$$\|E_\varepsilon(\lambda + \Lambda_0)^{-1} M_\varepsilon\|_{\mathcal{L}(C(\bar{\Omega}_\varepsilon), L^\infty(\Omega_\varepsilon))} \leqslant \frac{C}{|\lambda| + 1}, \tag{3.25}$$

and

$$\|E_\varepsilon \lambda(\lambda + \Lambda_0)^{-1} M_\varepsilon\|_{\mathcal{L}(C(\bar{\Omega}_\varepsilon), U^p_\varepsilon)} \leqslant C \tag{3.26}$$

where C is a constant that does not depend in ε.

We have now the following key result, which is analogous to Proposition 3.3 and Proposition 3.7

Proposition 3.10. *For $p, q > N$ and $f_\varepsilon \in U^p_\varepsilon$ we have*

$$\|\Lambda_\varepsilon^{-1} f_\varepsilon - E_\varepsilon \Lambda_0^{-1} M_\varepsilon f_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leqslant C(\varepsilon^{\frac{N}{q}} + \|W_\varepsilon - E_\varepsilon W_0 M_\varepsilon\|_{L^q(\Omega_\varepsilon)}) \, \|f_\varepsilon\|_{U^p_\varepsilon}. \tag{3.27}$$

with C independent of ε and f_ε.

Proof: Let $f_\varepsilon \in U^p_\varepsilon$ and let $u_\varepsilon = \Lambda_\varepsilon^{-1} f_\varepsilon = (A_\varepsilon + W_\varepsilon)^{-1} f_\varepsilon$. Consider the auxiliary function, $\tilde{u}_\varepsilon = (A_\varepsilon + E_\varepsilon W_0)^{-1} f_\varepsilon$, i.e.,

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon + W_\varepsilon u_\varepsilon &= f_\varepsilon, \quad \Omega_\varepsilon, \\ \quad \frac{\partial u_\varepsilon}{\partial n} &= 0, \quad \partial\Omega_\varepsilon. \end{cases} \tag{3.28}$$

$$\begin{cases} -\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon + W_0 \tilde{u}_\varepsilon &= f_\varepsilon, \quad \Omega_\varepsilon, \\ \quad \frac{\partial \tilde{u}_\varepsilon}{\partial n} &= 0, \quad \partial\Omega_\varepsilon. \end{cases} \tag{3.29}$$

From comparison results, it is easy to see that $|\tilde{u}_\varepsilon| \leqslant \bar{u}_\varepsilon$ where

$$\begin{cases} -\Delta \bar{u}_\varepsilon + \bar{u}_\varepsilon &= |f_\varepsilon|, \quad \Omega_\varepsilon, \\ \quad \frac{\partial u_\varepsilon}{\partial n} &= 0, \quad \partial\Omega_\varepsilon. \end{cases}$$

Applying Lemma A.11 of [3], we have that

$$\|\bar{u}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \|f_\varepsilon\|_{U^p_\varepsilon} \quad \text{for } p > N/2 \tag{3.30}$$

which implies
$$\|\tilde{u}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\|f_\varepsilon\|_{U^p_\varepsilon}.$$

Next, observe that
$$u_\varepsilon = (A_\varepsilon + E_\varepsilon W_0)^{-1} f_\varepsilon + (A_\varepsilon + E_\varepsilon W_0)^{-1}(E_\varepsilon W_0 - W_\varepsilon)u_\varepsilon$$
$$u_0 = (A_0 + W_0)^{-1} M_\varepsilon f_\varepsilon.$$

Hence,
$$\begin{aligned}
&\|u_\varepsilon - E_\varepsilon u_0\|_{L^q(\Omega_\varepsilon)}\\
&\quad\leqslant \|(A_\varepsilon + E_\varepsilon W_0)^{-1} - E_\varepsilon(A_0 + W_0)^{-1} M_\varepsilon f_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \|(A_\varepsilon + E_\varepsilon W_0)^{-1}(W_\varepsilon - E_\varepsilon W_0)u_\varepsilon\|_{L^q(\Omega_\varepsilon)}\\
&\quad\leqslant C\varepsilon^{\frac{N}{q}}\|f_\varepsilon\|_{U^p_\varepsilon} + \bar{C}\|(A_\varepsilon + E_\varepsilon W_0)^{-1}\|_{\mathcal{L}(L^q(\Omega_\varepsilon))}\|W_\varepsilon - E_\varepsilon W_0\|_{L^q(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\\
&\quad\leqslant \tilde{C}(\varepsilon^{\frac{N}{q}} + \|W_\varepsilon - E_\varepsilon W_0\|_{L^q(\Omega_\varepsilon)})\|f_\varepsilon\|_{U^p_\varepsilon}.
\end{aligned}$$
where we have used (3.20) and the fact that that there is a constant C, independent of ε and of $q \in [1, \infty]$, such that $\|(A_\varepsilon + W_0)^{-1}\|_{\mathcal{L}(L^q(\Omega_\varepsilon))} \leqslant C$. This shows the lemma. ■

As an immediate corollary, we have

Corollary 3.11. *For $p, q > N$ we have*
$$\varepsilon^{-\frac{N-1}{q}}\|\Lambda_\varepsilon^{-1} - E_\varepsilon\Lambda_0^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \to 0 \text{ as } \varepsilon \to 0. \tag{3.31}$$

Proof: We just need to apply the previous proposition and hypothesis (**H**). ■

Now consider a compact subset K of the complex plane which is contained in the resolvent set of the operator Λ_0. Let $c(K)$ be a positive constant such that
$$\sup_{\lambda \in K}\|(\lambda + \Lambda_0)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \leqslant c(K).$$
Also, let $\Sigma_\theta := \{z \in C : |\arg(z)| \leqslant \pi - \theta\}$, for $0 < \theta < \pi/2$.

Proposition 3.12. *For $p, q > N$, there exists a constant $C = C(K, \theta)$, a number $\varepsilon_0 > 0$ and a function $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for each $\lambda \in K \cup \Sigma_\theta$ and $0 < \varepsilon \leqslant \varepsilon_0$ we have*
$$\varepsilon^{-\frac{N-1}{q}}\|(\lambda + \Lambda_\varepsilon)^{-1} - E_\varepsilon(\lambda + \Lambda_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \leqslant C\,(1 + |\lambda|^{1-\alpha})\eta(\varepsilon), \tag{3.32}$$
where $0 < \alpha < 1 - \frac{N}{2p} < 1$

Proof: Observe first that the spectrum of the operators Λ_ε and Λ_0 are subsets of $[1, +\infty)$. Hence, if $\lambda \in \Sigma_\theta$ both $(\lambda + \Lambda_\varepsilon)^{-1}$ and $(\lambda + \Lambda_0)^{-1}$ make perfect sense for $0 < \varepsilon \leqslant \varepsilon_0$.

Moreover, by the compact convergence of $A_\varepsilon^{-1} \to A_0^{-1}$, the convergence of $W_\varepsilon \to W_0$ and since $\|(\lambda + \Lambda_0)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} = \|(\lambda + V_0 + A_0)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \leqslant c(K)$ for each $\lambda \in K$ which is a compact set in C, we have that $(\lambda + A_\varepsilon + V_\varepsilon)$ and $(\lambda + A_\varepsilon + V_0)$ are invertible for $0 < \varepsilon < \varepsilon_0$ and $\lambda \in \Lambda_0$ and $\|(\lambda + \Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \leqslant \tilde{c}(K)$, for some constant $\tilde{c}(K)$ and for all $\lambda \in K$. If this is not the case, then we could get a sequence of $\varepsilon_n \to 0$ and $\lambda_n \to \tilde{\lambda} \in K$ such that $\|(\lambda_n + \Lambda_{\varepsilon_n})^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \to +\infty$. But this is in contradiction with the compact convergence of $(\lambda_n + \Lambda_{\varepsilon_n})^{-1}$ to $(\tilde{\lambda} + \Lambda_0)^{-1}$, see Lemma 4.7 of [3].

Hence, with this argument and with (3.22) and (3.24) we obtain

$$\|\lambda\,(\lambda+\Lambda_\varepsilon)^{-1}\,\|_{\mathcal{L}(L^q(\Omega_\varepsilon))} \leqslant C, \quad \text{for} \quad \lambda \in K \cup \Sigma_\theta, \tag{3.33}$$

$$\|E_\varepsilon \lambda(\lambda+\Lambda_0)^{-1} M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^p_\varepsilon)} \leqslant C(1+|\lambda|^{1-\alpha}), \quad \text{for} \quad \lambda \in K \cup \Sigma_\theta. \tag{3.34}$$

with $0 < \alpha < 1 - \frac{N}{2p} < 1$. Applying Lemma 3.5 with Λ_0 in place of A_0 and λ in place of V_0, we have

$$\begin{aligned}
&\|(\lambda+\Lambda_\varepsilon)^{-1} - E_\varepsilon(\lambda+\Lambda_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \leqslant \\
&\quad \|I+\lambda(\lambda+\Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(L^q(\Omega_\varepsilon))}\|\Lambda_\varepsilon^{-1} - E_\varepsilon\Lambda_0^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))}\|[I - E_\varepsilon\lambda(\lambda+\Lambda_0)^{-1}M_\varepsilon]\|_{\mathcal{L}(U^p_\varepsilon)} \\
&\quad \leqslant C(1+|\lambda|^{1-\alpha})\|\Lambda_\varepsilon^{-1} - E_\varepsilon\Lambda_0^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \leqslant C\varepsilon^{\frac{N-1}{q}}(1+|\lambda|^{1-\alpha})\eta(\varepsilon)
\end{aligned}$$

where $\eta(\varepsilon) = \varepsilon^{-\frac{N-1}{q}}\,\|\Lambda_\varepsilon^{-1} - E_\varepsilon\Lambda_0^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \to 0$ as $\varepsilon \to 0$ by Corollary 3.11. This proves the proposition. ■

Remark 3.13. *The results of Proposition* 3.12 *also hold for the operator A_ε instead of Λ_ε, that is with $W_\varepsilon = W_0 = 0$.*

Corollary 3.14. *In the conditions of Proposition* 3.12*, we have the following estimates,*

$$\|(\lambda+\Lambda_\varepsilon)^{-1} - E_\varepsilon(\lambda+\Lambda_0)^{-1}M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C\,(1+|\lambda|^{1-\alpha})\eta(\varepsilon), \tag{3.35}$$

$$\|(\lambda+\Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C(1+|\lambda|^{1-\alpha}) \tag{3.36}$$

Proof: To prove (3.35) we just use that $\varepsilon^{-\frac{N-1}{q}}\,\|\cdot\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} \leqslant \|\cdot\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)}$ in (3.32). To prove (3.36) we just use (3.35) and (3.24), to obtain

$$\|(\lambda+\Lambda_\varepsilon)^{-1}\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C(1+|\lambda|^{1-\alpha})\eta(\varepsilon) + \frac{C}{|\lambda|^\alpha+1} \leqslant C(1+|\lambda|^{1-\alpha}),$$

as we wanted to show. ■

These results play a fundamental role on the convergence of the linear semigroups for it will ensure the uniform convergence of the integrals defining them and will allow us to pass to the limit.

3.3. **Rate of convergence of hyperbolic equilibria and of its linearizations.** In this Subsection we will obtain rates of convergence of hyperbolic equilibria which, besides being interesting by themselves, they show that if we consider the potentials $V_\varepsilon = -f'(u^*_\varepsilon)$, $V_0 = -f'(u^*_0)$ then hypothesis (**H**) from Subsection 3.2 is satisfied, with $a = \sup\{|f'(s)| : s \in \mathbb{R}\}$. This will ensure that all the results from Subsection 3.2 apply for $\Lambda_\varepsilon = A_\varepsilon - f'(u^*_\varepsilon) + a$ and $\Lambda_0 = A_0 - f'(u^*_0) + a$.

Proposition 3.15. *Let u^*_0 be a hyperbolic equilibrium for* (1.2) *and (from the results in* [3]*) let u^*_ε be the sequence of hyperbolic equilibria for* (1.1) *satisfying that u^*_ε E-converges to u^*_0. Then, for $q > N$, we have*

$$\|u^*_\varepsilon - E_\varepsilon u^*_0\|_{L^q(\Omega)} \leqslant C\varepsilon^{\frac{N}{q}} \tag{3.37}$$

and

$$\varepsilon^{-\frac{N-1}{q}}\|u^*_\varepsilon - E_\varepsilon u^*_0\|_{U^p_\varepsilon} \to 0, \qquad \text{as } \varepsilon \to 0. \tag{3.38}$$

Proof: Let $u_0^* = (w_0^*, v_0^*)$ be a hyperbolic equilibrium point for (1.2) and u_ε^* an equilibrium point for (1.1) with $\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{U_\varepsilon^p} \xrightarrow{\varepsilon\to 0} 0$. For $V_0(x) = -f'(u_0^*(x))$, we write

$$u_\varepsilon^* = (A_\varepsilon + V_0)^{-1}(f(u_\varepsilon^*) + V_0 u_\varepsilon^*) \text{ and } u_0^* = (A_0 + V_0)^{-1}(f(u_0^*) + V_0 u_0^*).$$

Hence, taking norms in $L^q(\Omega)$, we get

$$\begin{aligned}
\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^q(\Omega)} &= \|(A_\varepsilon + V_0)^{-1}(f(u_\varepsilon^*) + V_0 u_\varepsilon^*) - E_\varepsilon(A_0 + V_0)^{-1}(f(u_0^*) + V_0 u_0^*)\|_{L^q(\Omega)} \\
&\leqslant \|(A_\varepsilon + V_0)^{-1} - E_\varepsilon(A_0 + V_0)^{-1} M_\varepsilon)\,(f(u_\varepsilon^*) + V_0 u_\varepsilon^*)\|_{L^q(\Omega)} \\
&+ \|E_\varepsilon(A_0 + V_0)^{-1} M_\varepsilon\,[f(u_\varepsilon^*) - V_0 u_\varepsilon^* - E_\varepsilon(f(u_0^*) + V_0 M_\varepsilon E_\varepsilon u_0^*)]\|_{L^q(\Omega)} \\
&\leqslant C\,\varepsilon^{N/q}\|f(u_\varepsilon^*) + V_0 u_\varepsilon^*\|_{L^q(\Omega_\varepsilon)} \\
&+ \|E_\varepsilon(A_0 + V_0)^{-1} M_\varepsilon[f(u_\varepsilon^*) - E_\varepsilon f(u_0^*) - V_0(u_\varepsilon^* - E_\varepsilon u_0^*)]\|_{L^q(\Omega)} \\
&\leqslant C\,\varepsilon^{N/q} + \|E_\varepsilon(A_0 + V_0)^{-1} M_\varepsilon z_\varepsilon\|_{L^q(\Omega)}.
\end{aligned}$$

where $z_\varepsilon = f(u_\varepsilon^*) - f(u_0^*) + V_0(u_\varepsilon^* - u_0^*)$ and we have used Proposition 3.7, the boundedness of f' and that u_ε^* is also bounded in the sup norm uniformly in ε.

We have

$$\begin{aligned}
|z_\varepsilon(x)| &= |f(u_\varepsilon^*(x)) - f(u_0^*(x)) + f'(E_\varepsilon u_0^*(x))(u_\varepsilon^*(x) - E_\varepsilon u_0^*(x))| \\
&\leqslant |[f'(\chi_\varepsilon^*(x)) - f'(E_\varepsilon u_0^*(x))]\,(u_\varepsilon^*(x) - E_\varepsilon u_0^*(x))|
\end{aligned}$$

where $\chi_\varepsilon^*(x) = \theta(x) u_\varepsilon^*(x) + (1 - \theta(x)) E_\varepsilon u_0^*(x)$ and $0 \leqslant \theta(x) \leqslant 1$, $x \in \Omega_\varepsilon$.

Using that $|f'(\cdot)| \leqslant C$ we have,

$$\|z_\varepsilon\|_{L^r(\Omega)} \leqslant C\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^r(\Omega)}, \quad \forall 1 \leqslant r \leqslant +\infty.$$

Also,

$$\|z_\varepsilon\|_{L^r(\Omega)} \leqslant \|f'(\chi_\varepsilon^*(x)) - f'(E_\varepsilon u_0^*(x))\|_{L^s(\Omega)}\,\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^t(\Omega)}, \quad \frac{1}{r} = \frac{1}{s} + \frac{1}{t}$$

But

$$\begin{aligned}
&\|f'(\chi_\varepsilon^*(x)) - f'(E_\varepsilon u_0^*(x))\|_{L^\infty(\Omega)} \leqslant C \\
&\|f'(\chi_\varepsilon^*(x)) - f'(E_\varepsilon u_0^*(x))\|_{L^1(\Omega)} \leqslant C\|\chi_\varepsilon^*(x)) - E_\varepsilon u_0^*(x)\|_{L^1(\Omega)} \leqslant C\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^1(\Omega)}.
\end{aligned}$$

Hence, using interpolation $\|f'(\chi_\varepsilon^*(x)) - f'(E_\varepsilon u_0^*(x))\|_{L^s(\Omega)} \leqslant C\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^1(\Omega)}^{1/s}$. So

$$\|z_\varepsilon\|_{L^r(\Omega)} \leqslant C\,\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^1(\Omega)}^{1/s}\,\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^t(\Omega)} \leqslant C\,\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^t(\Omega)}^{1+\frac{1}{s}}$$

But if we define $w_\varepsilon = E_\varepsilon(A_0 + B)^{-1} M_\varepsilon z_\varepsilon$, we know from (3.16) that

$$\|w_\varepsilon\|_{L^q(\Omega)} \leqslant C\|z_\varepsilon\|_{L^r(\Omega)} \text{ for some } r < q.$$

Hence we can choose $\dfrac{1}{r} = \dfrac{1}{s} + \dfrac{1}{q}$ $(t = q, \dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{q} > 0)$. So

$$\|E_\varepsilon(A_0 + B)^{-1} M_\varepsilon z_\varepsilon\|_{L^q(\Omega)} \leqslant C\|z_\varepsilon\|_{L^r(\Omega)} \leqslant C\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)}^{1+\frac{1}{r}-\frac{1}{q}}.$$

Hence

$$\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)} \leqslant C\varepsilon^{N/q} + C\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)}^{1+\frac{1}{r}-\frac{1}{q}}.$$

Since we know that $\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)} \to 0$ (since $\|u_\varepsilon^* - u_0^*\|_{U_\varepsilon^p} \to 0$ as $\varepsilon \to 0$) then $\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)} \leqslant C\,\varepsilon^{N/q}$, which shows the first statement of the lemma. For the second one, we just realize that

$$\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega)} + \|u_\varepsilon^* - u_0^*\|_{L^q(R_\varepsilon)} \leqslant C\,\varepsilon^{N/q} + C\,\mathbf{o}(\varepsilon^{\frac{N-1}{q}}) = \mathbf{o}(\varepsilon^{\frac{N-1}{q}}).$$

That is,

$$\varepsilon^{-\frac{N-1}{q}}\,\|u_\varepsilon^* - u_0^*\|_{L^q(\Omega_\varepsilon)} \to 0 \,\mathrm{as}\,\varepsilon \to 0.$$

■

Corollary 3.16. *In the conditions of Proposition* 3.15*, if we denote by* $V_\varepsilon = -f'(u_\varepsilon^*)$*,* $V_0 = -f'(u_0^*)$ *and* $a = \sup\{|f'(s)|;\, s \in \mathbb{R}\}$*, then hypothesis* (***H***) *from Subsection* 3.2 *is satisfied. Hence, all the results of that Subsection can be applied to the case where the potentials are given by* $V_\varepsilon = -f'(u_\varepsilon^*)$ *and* $V_0 = -f'(u_0^*)$*.*

Proof: Since

$$\|V_\varepsilon - E_\varepsilon V_0\|_{L^q(\Omega_\varepsilon)} = \|f'(u_\varepsilon^*) - E_\varepsilon f'(u_0^*)\|_{L^q(\Omega_\varepsilon)} \leqslant \|f''\|_{L^\infty(\mathbb{R})}\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{L^q(\Omega_\varepsilon)} = \mathbf{o}(\varepsilon^{\frac{N-1}{q}}),$$

the result follows. ■

4. Convergence of Linear Semigroups

In this section we analyze the convergence properties of the linear semigroups generated by the operators $A_\varepsilon + V_\varepsilon$, $A_0 + V_0$ where the potentials V_ε, V_0 satisfy hypothesis (**H**) from Subsection 3.2. Later on we will be interested in applying the results from this section to the semigroups generated by A_ε, A_0 and also by $A_\varepsilon - f'(u_\varepsilon^*)$ and $A_0 - f'(u_0^*)$, where u_ε^*, u_0^* are hyperbolic equilibria of the perturbed and limit problem respectively.

As in Section 3.2, let $W_\varepsilon = V_\varepsilon + a > 00$, $W_0 = V_0 + a > 0$, (see hypothesis (**H**)) and $\Lambda_\varepsilon = A_\varepsilon + W_\varepsilon$, $\Lambda_0 = A_0 + W_0$.

As we have already seen in [4], the operators $-A_0$, $-(A_0 + V_0)$ and $-\Lambda_0$ do not generate strongly continuous semigroups in U_0^p. Nonetheless they generate certain singular semigroups as we briefly recall.

Let $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and let Γ be the boundary of Σ_θ oriented such that the imaginary part grows as λ runs in Γ. Notice that the semigroups generated by $-\Lambda_0$ and by $-(A_0 + V_0)$ are related by a multiplicative factor of the form e^{at}.

Proceeding as in [4] we define

$$e^{-\Lambda_0 t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda + \Lambda_0)^{-1}\,d\lambda, \quad t > 0. \tag{4.1}$$

Then, $e^{-\Lambda_0 t}$ satisfies the semigroup properties but strong continuity fails at $t = 0$ for data which are not sufficiently smooth. Nonetheless, several of the properties of analytic semigroup will still hold for sufficiently regular data. We say that $\{e^{-\Lambda_0 t} : t \geqslant 0\}$ is the semigroup generated by $-\Lambda_0$ and do not make any allusion to continuity. We refer to [4] for a detailed study of the semigroup generated by $-\Lambda_0$.

In what follows we recall some simple properties of the semigroup $\{e^{-\Lambda_0 t} : t \geqslant 0\}$ that we will employed later in this paper.

The next result investigates the singularity of $\{E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon : t > 0\}$ at $t = 0$ in $\mathcal{L}(U^p_\varepsilon)$. Its proof is a consequence of Proposition 3.12 and (4.1).

Lemma 4.1. *For any $p \geqslant q > \frac{N}{2}$ and for $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$, there is a constant C, independent of ε, such that*

$$\|E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon u\|_{U^p_\varepsilon} \leqslant C t^{\alpha - 1} \|u\|_{U^q_\varepsilon}, \quad t > 0, \quad u \in U^q_\varepsilon, \tag{4.2}$$

and

$$\|E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon u\|_{U^p_\varepsilon} \leqslant C \|u\|_{U^\infty_\varepsilon}, \quad t > 0, \quad u \in U^\infty_\varepsilon. \tag{4.3}$$

From Lemma 3.8 it follows that, $-\Lambda_\varepsilon$ generates an analytic semigroup $\left\{e^{-\Lambda_\varepsilon t} : t \geqslant 0\right\}$ in U^p_ε given by

$$e^{-\Lambda_\varepsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \, (\lambda + \Lambda_\varepsilon)^{-1} \, d\lambda, \quad t > 0, \tag{4.4}$$

where $\Gamma \subset \rho(-\Lambda_\varepsilon)$ is the boundary of $\Sigma_\theta\}$ oriented such that the imaginary part grows as λ runs in Γ. Note that Γ is independent of ε. It follows from (3.22), (3.23) and (4.4) that the following estimates hold

$$\|e^{-\Lambda_\varepsilon t} w\|_{U^p_\varepsilon} \leqslant C \varepsilon^{\frac{-N+1}{p}} \|w\|_{U^p_\varepsilon}, \quad t \geqslant 0, \ w \in U^p_\varepsilon, \tag{4.5}$$

$$\|e^{-\Lambda_\varepsilon t} w\|_{L^p(\Omega_\varepsilon)} \leqslant C \|w\|_{L^p(\Omega_\varepsilon)}, \quad t \geqslant 0, \ w \in L^p(\Omega_\varepsilon), \tag{4.6}$$

and

$$\|e^{-\Lambda_\varepsilon t} w\|_{U^p_\varepsilon} \leqslant C \|w\|_{U^\infty_\varepsilon}, \quad t \geqslant 0, \ w \in U^\infty_\varepsilon, \tag{4.7}$$

for some constant $C > 0$ that does not depend on ε. That is, the linear semigroup $e^{\Lambda_\varepsilon t}$ is bounded in $\mathcal{L}(L^p(\Omega_\varepsilon))$ uniformly with respect to ε.

We analyze now the convergence properties of the semigroups. To accomplish this task we will use extensively the resolvent estimates of the previous section applied to the integral expression of the semigroup.

Proposition 4.2. *There are $\gamma > 0$, $\beta \in \mathbb{R}$, $p, q > N$ and function $\rho : [0,1] \to [0, \infty)$ with $\rho(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$ such that*

$$\|e^{(A_\varepsilon + V_\varepsilon)t} - E_\varepsilon e^{(A_0 + V_0)t} M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C \, e^{\beta t} t^{-\gamma} \rho(\varepsilon), \quad t > 0. \tag{4.8}$$

Proof: Observe first that $e^{-(A_\varepsilon + V_\varepsilon)t} - E_\varepsilon e^{-(A_0 + V_0)t} M_\varepsilon = e^{at}(e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon)$, so that it is sufficient to prove an estimate of the type (4.8) for the difference $e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon$.

Since,

$$e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon = \frac{1}{2\pi i} \int\limits_\Gamma ((\lambda + \Lambda_\varepsilon)^{-1} - E_\varepsilon (\lambda + \Lambda_0)^{-1} M_\varepsilon) e^{\lambda t} d\lambda, \tag{4.9}$$

it follows from Proposition 3.12 that

$$\begin{aligned} \varepsilon^{-\frac{N-1}{q}} \|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, L^q(\Omega_\varepsilon))} &\leqslant \frac{C}{2\pi} \left| \int\limits_\Gamma (1 + |\lambda|^{1-\alpha}) |e^{\lambda t}| d\lambda \right| \eta(\varepsilon) \\ &\leqslant C \, t^{-(2-\alpha)} \eta(\varepsilon) \end{aligned}$$

and consequently

$$\|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C\, t^{-(2-\alpha)}\eta(\varepsilon).$$

On the other hand, by comparison (maximum principle) we have

$$\|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^\infty_\varepsilon)} \leqslant \|e^{-\Lambda_\varepsilon t}\|_{\mathcal{L}(U^\infty_\varepsilon)} + \|E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^\infty_\varepsilon)} \leqslant C.$$

Noting that $\|\cdot\|_{U^q_\varepsilon} \leqslant c\|\cdot\|_{U^\infty_\varepsilon}$ for some $c>0$ independent of ε, it follows that

$$\|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^\infty_\varepsilon, U^q_\varepsilon)} \leqslant C.$$

By interpolation (see [8, Theorem 6.27])

$$\|e^{-\Lambda_\varepsilon} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^{\bar p}_\varepsilon, U^q_\varepsilon)} \leqslant C\, t^{-\theta(2-\alpha)}\eta^\theta(\varepsilon).$$

where $p \leqslant \bar p < \infty$ and $0 \leqslant \theta \leqslant 1$. Taking θ small we can make $\theta(2-\alpha) < 1$.

That is

$$\|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^{\bar p}_\varepsilon, U^q_\varepsilon)} \leqslant C\, t^{-\gamma}\eta(\varepsilon)^\theta,\ \gamma < 1.$$

Hence, if we define $\rho(\varepsilon) = \eta(\varepsilon)^\theta$, we have

$$\|e^{(A_\varepsilon+V_\varepsilon)t} - E_\varepsilon e^{(A_0+V_0)t} M_\varepsilon\|_{\mathcal{L}(U^{\bar p}_\varepsilon, U^q_\varepsilon)} = e^{at}\|e^{-\Lambda_\varepsilon t} - E_\varepsilon e^{-\Lambda_0 t} M_\varepsilon\|_{\mathcal{L}(U^{\bar p}_\varepsilon, U^q_\varepsilon)} \leqslant C\, e^{at} t^{-\gamma}\rho(\varepsilon)$$

which shows the result with $\rho(\varepsilon) = \eta(\varepsilon)^\theta$ and $\beta = a$. ■

Let us consider now a real number b with the property that there exists a $\delta > 0$, small, such that $[b-\delta, b+\delta] \cap \sigma(-(A_0+V_0)) = \emptyset$. That is, the spectrum of the operator $-(A_0+V_0)$, which is all real, is divided in two parts, σ_0^+ which is above $b+\delta$ and it is a finite set and σ_0^- which is below $b-\delta$ and it is an infinite set (a sequence that goes to $-\infty$). From the continuity properties of the spectrum, (see [3]) we have that for ε small enough $[b-\delta, b+\delta] \cap \sigma(-(A_\varepsilon+V_\varepsilon)) = \emptyset$ and the spectra of $-(A_\varepsilon+V_\varepsilon)$, which is also real, is divided in two parts σ_ε^+, above $b+\delta$ and σ_ε^-, below $b-\delta$. Moreover, we can choose a fixed closed curve $\Gamma_b^+ \subset \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geqslant b+\delta\}$ which encloses σ_ε^+ for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$ for some ε_0 small. Moreover, we denote by $\Gamma_b^- = \{z \in \mathbb{C} : \arg(z-(b-\delta)) = \pi - \theta\}$ for some $0 < \theta < \pi/2$.

We decompose U^p_ε using the projection

$$Q_\varepsilon^+ = Q(\sigma_\varepsilon^+) = \frac{1}{2\pi i}\int_{\Gamma_b^+} (\lambda + A_\varepsilon + V_\varepsilon)^{-1} d\lambda. \tag{4.10}$$

Proposition 4.3. *For $p, q > N$ large enough, we have that there are constants $C > 0$, $\gamma < 1$, independent of ε and a function $\rho(\varepsilon)$, with $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for $t > 0$*

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+)) - E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C\, e^{bt} t^{-\gamma}\rho(\varepsilon) \tag{4.11}$$

$$\|E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C\, e^{bt} t^{-\gamma} \tag{4.12}$$

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+))\|_{\mathcal{L}(U^p_\varepsilon, U^q_\varepsilon)} \leqslant C\, e^{bt} t^{-\gamma} \tag{4.13}$$

Proof: We have

$$e^{-(A_0+V_0)t}(I - Q(\sigma_0^+)) = \frac{1}{2\pi i}\int_{\Gamma_b^-} (\lambda + A_0 + V_0(x))^{-1} e^{\lambda t} d\lambda.$$

Plugging norms and using estimate (3.5) we get

$$\|e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))\|_{\mathcal{L}(U_0^p,U_0^q)} \leqslant \left|\frac{1}{2\pi}\int_{\Gamma_b^-}\frac{|e^{\lambda t}|}{|1+|\lambda|^{1-\alpha}}d\lambda\right|$$

and elementary integration shows

$$\|e^{(A_0+V_0)t}(I-Q(\sigma_0^+))\|_{\mathcal{L}(U_0^p,U_0^q)} \leqslant Ce^{bt}t^{-\alpha} \tag{4.14}$$

which shows (4.12) with $\gamma=\alpha$.

In a similar way,

$$e^{-(A_\varepsilon+V_\varepsilon)t}(I-Q(\sigma_\varepsilon^+))-E_\varepsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\varepsilon = \frac{1}{2\pi i}\int_{\Gamma_b^-}((\lambda+A_\varepsilon+V_\varepsilon(x))^{-1}-E_\varepsilon(\lambda+A_0+V_0(x))^{-1}M_\varepsilon)e^{\lambda t}d\lambda.$$

So

$$\begin{aligned}\|e^{-(A_\varepsilon+V_\varepsilon)t}(I-Q(\sigma_\varepsilon^+))-E_\varepsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant \\ \frac{1}{2\pi}\left|\int_{\Gamma_b^-}|e^{\lambda t}|\|(\lambda+A_\varepsilon+V_\varepsilon(x))^{-1}-E_\varepsilon(\lambda+A_0+V_0(x))^{-1}M_\varepsilon)\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)}d\lambda\right| \\ \leqslant \frac{1}{2\pi}\int_{\Gamma_b^-}|e^{\lambda t}|(1+|\lambda|^{1-\alpha})d\lambda\eta(\varepsilon)d\lambda \\ \leqslant \frac{C}{2\pi}e^{bt}t^{-(2-\alpha)}\eta(\varepsilon),\end{aligned}$$

where we have applied Proposition 3.12. Therefore,

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I-Q(\sigma_\varepsilon^+))-E_\varepsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant C\,e^{bt}t^{-(2-\alpha)}\eta(\varepsilon). \tag{4.15}$$

This estimate does not show yet the proposition since the exponent $2-\alpha>1$. We will do an interpolation argument to conclude with the correct estimate. For this, let us see now that $Q(\sigma_\varepsilon^+):U_\varepsilon^p\to U_\varepsilon^p$ satisfies $\|Q(\sigma_\varepsilon^+)\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^p)}\leqslant C$ independent of ε. To see this, just observe that

$$Q(\sigma_\varepsilon^+)=\frac{1}{2\pi i}\int_{\Gamma_b^+}(\lambda+A_\varepsilon+V_\varepsilon)^{-1}d\lambda.$$

Applying now the estimate of Propostion 3.12, we obtain that

$$\|(\lambda+A_\varepsilon+V_\varepsilon)^{-1}\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^p)}\leqslant C$$

for $\lambda\in\Gamma_b^-$ and with C independent of ε. From this last expresion and using the boundedness of Γ_b^- we get $\|Q(\sigma_\varepsilon^+)\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^p)}\leqslant C$, for all $0\leqslant\varepsilon\leqslant 1$.

Moreover, for the limit semigroup and for $0 < t \leqslant 1$, we obtain from (4.14)

$$\|E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant Ct^{-\alpha}.$$

Hence for $0 < t \leqslant 1$, we get that

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+))\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant \|e^{-(A_\varepsilon+V_\varepsilon)t}\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)}(1 + \|Q(\sigma_\varepsilon^+)\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^p)})$$
$$C\left(\|e^{-(A_\varepsilon+V_\varepsilon)t} - E_\varepsilon e^{-(A_0+V_0)t}M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} + \|E_\varepsilon e^{-(A_0+V_0)t}M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)}\right) \leqslant C(t^{-\gamma} + t^{-\alpha+1})$$

where we are using the bounds given by Proposition 4.2

Hence, for $0 < t \leqslant 1$

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+)) - E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant Ct^{-\bar{\gamma}} \tag{4.16}$$

where $\bar{\gamma} = \max\{\gamma, 1 - \alpha\}$.

Interpolating (4.15) and (4.16) we obtain, for $0 < t \leqslant 1$,

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+)) - E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant \tag{4.17}$$
$$(Ct^{-(2-\alpha)}\eta(\varepsilon))^\theta(Ct^{-\bar{\gamma}})^{1-\theta} \leqslant Ct^{-(2-\alpha)\theta-(1-\theta)\bar{\gamma}}\eta(\varepsilon)^\theta \tag{4.18}$$

where we have used that $e^{bt} \leqslant C$ for $0 \leqslant t \leqslant 1$. Choosing $\theta > 0$ small enough so that $(2-\alpha)\theta + (1-\theta)\bar{\gamma} < 1$, we obtain the estimate for $0 < t \leqslant 1$.

Now for $t \geqslant 1$, from (4.15) we get

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}(I - Q(\sigma_\varepsilon^+)) - E_\varepsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant Ce^{bt}\eta(\varepsilon)$$

Putting together both estimates, we prove (4.11). To prove (4.13) we just use (4.11) and (4.12). This concludes the proof of the proposition. ■

We also have

Corollary 4.4. *For the case $V_\varepsilon = V_0 \equiv 0$ and with $b \in (-1, 0)$ a fixed number, we have that $Q(\sigma_\varepsilon^+) \equiv 0$ for ε small enough and we have*

$$\|e^{-A_\varepsilon t} - E_\varepsilon e^{-A_0 t}M_\varepsilon\|_{\mathcal{L}(U_\varepsilon^p,U_\varepsilon^q)} \leqslant C\, e^{bt}t^{-\gamma}\rho(\varepsilon).$$

Remark 4.5. *Observe that we can consider the case where $V_0 = -f'(u_0^*)$, $V_\varepsilon = -f'(u_\varepsilon^*)$ with u_0^* and u_ε^* hyperbolic equilibria satisfying u_ε^* converging to u_0^* (see [3]). In this case, we can always apply Proposition 4.3 with $b < 0$, a number dividing the spectrum among the stable part, that is with negative real part, and the unstable spectrum, that is with positive real part.*

Let us conclude the section with the following useful unifom estimates of the semigroup on the linear unstable manifold

Proposition 4.6. *There are constants $C \geqslant 1$ and $\beta > 0$ such that*

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}Q_\varepsilon^+\|_{\mathcal{L}(U_\varepsilon^q,U_\varepsilon^p)} \leqslant Ce^{\beta t},\ t \leqslant 0 \tag{4.19}$$

Proof: Observe that

$$e^{-(A_\varepsilon+V_\varepsilon)t}Q_\varepsilon^+ = \int_{\Gamma^+} e^{\lambda t}(\lambda + A_\varepsilon + V_\varepsilon)^{-1}d\lambda.$$

Using (3.36) and noticing that the curve Γ^+ is bounded, we have

$$\|e^{-(A_\varepsilon+V_\varepsilon)t}Q_\varepsilon^+\|_{\mathcal{L}(U_\varepsilon^q,U_\varepsilon^p)} \leqslant C\left|\int_{\Gamma^+}|e^{\lambda t}|d\lambda\right| \leqslant Ce^{\beta t}$$

which shows the result. ■

5. Continity of nonlinear semigroups and upper semicontinuity of attractors

Now that we have obtained in the previous section the continuity of linear semigroups we proceed to obtain the continuity of nonlinear semigroups using the Variation of Constants Formula. After we obtain the continuity of nonlinear semigroups we will proceed to obtain the upper semicontinuity of the family of attractors $\{\mathcal{A}_\varepsilon : \varepsilon \in [0,1]\}$.

To this end we will follow the ideas in [1] that relate the continuity of the linear semigroups with the continuity of the nonlinear semigroups for dissipative parabolic equations by using the variation of constants formula. This in turn will imply the upper semicontinuity of the attractors and the stationary states.

For $\varepsilon \in [0,1]$, let $\{T_\varepsilon(t) : t \geqslant 0\}$ be the semigroups defined in U_ε^p by the variation of constants formula

$$T_\varepsilon(t,u_\varepsilon) = e^{-A_\varepsilon t}u_\varepsilon + \int_0^t e^{-A_\varepsilon(t-s)}f_\varepsilon(T_\varepsilon(s,u_\varepsilon))ds. \tag{5.1}$$

If $\mathcal{E}_\varepsilon$ denotes the set of stationary states (2.6), $\varepsilon \in [0,\varepsilon_0]$, it has been obtained in [3, Section 5] that, $\{\mathcal{E}_\varepsilon : \varepsilon \in [0,\varepsilon_0]\}$ is upper semicontinuous at $\varepsilon = 0$ in U_ε^p; that is,

$$\sup_{u_\varepsilon^*\in\mathcal{E}_\varepsilon}\left[\inf_{u_0^*\in\mathcal{E}_0}\{\|u_\varepsilon^* - E_\varepsilon u_0^*\|_{U_\varepsilon^p}\}\right] \to 0, \quad \text{as } \varepsilon \to 0 \tag{5.2}$$

We are now in position to prove the following result

Proposition 5.1. *There exists a $0 \leqslant \gamma < 1$ and a function $c(\varepsilon)$ with $c(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ such that, for each $\tau > 0$ we have*

$$\|T_\varepsilon(t,u_\varepsilon) - E_\varepsilon T_0(t, M_\varepsilon u_\varepsilon)\|_{U_\varepsilon^p} \leqslant M(\tau)c(\varepsilon)t^{-\gamma}, \quad t \in (0,\tau], \quad u_\varepsilon \in \mathcal{A}_\varepsilon, \quad \varepsilon \in (0,\varepsilon_0]. \tag{5.3}$$

Moreover, the family of attractors $\{\mathcal{A}_\varepsilon : \varepsilon \in [0,\varepsilon_0]\}$ is upper semicontinuous at $\varepsilon = 0$ in U_ε^p, in the sense that

$$\sup_{u_\varepsilon\in\mathcal{A}_\varepsilon}\left[\inf_{u_0\in\mathcal{A}_0}\{\|u_\varepsilon - E_\varepsilon u_0\|_{U_\varepsilon^p}\}\right] \to 0, \quad \textit{as } \varepsilon \to 0 \tag{5.4}$$

Proof: To prove this result we follow [1, 7]. Notice that the nonlinear semigroups $T_\varepsilon(t)$ are given by (5.1). Hence, estimating $T_\varepsilon(t,u_\varepsilon) - E_\varepsilon T_0(t, M_\varepsilon u_\varepsilon)$ and with some elementary computations we obtain

$$\begin{aligned}\|T_\varepsilon(t,u_\varepsilon) - E_\varepsilon T_0(t,M_\varepsilon u_\varepsilon)\|_{U_\varepsilon^p} &\leqslant \|e^{-A_\varepsilon t}u_\varepsilon - E_\varepsilon e^{-A_0 t}M_\varepsilon u_\varepsilon\|_{U_\varepsilon^p} \\ &+ \int_0^t \|\left(e^{-A_\varepsilon t} - E_\varepsilon e^{-A_0 t}M_\varepsilon\right)f_\varepsilon(T_\varepsilon(s,u_\varepsilon))\|_{U_\varepsilon^p}\,ds \\ &+ \int_0^t \|E_\varepsilon e^{-A_0 t}(M_\varepsilon f_\varepsilon(T_\varepsilon(s,u_\varepsilon)) - f_0(T_0(s,M_\varepsilon u_\varepsilon)))\|_{U_\varepsilon^p}\,ds, \quad \varepsilon \in [0,\varepsilon_0].\end{aligned}$$

Note that

$$\int_0^t \|E_\varepsilon e^{-A_0 t}(M_\varepsilon f_\varepsilon(T_\varepsilon(s,u_\varepsilon)) - f_0(T_0(s,M_\varepsilon u_\varepsilon)))\|_{U_\varepsilon^p}\,ds$$
$$= \int_0^t \|E_\varepsilon e^{-A_0 t}(M_\varepsilon f_\varepsilon(T_\varepsilon(s,u_\varepsilon)) - M_\varepsilon E_\varepsilon f_0(T_0(s,M_\varepsilon u_\varepsilon)))\|_{U_\varepsilon^p}\,ds$$
$$= \int_0^t \|E_\varepsilon e^{-A_0 t} M_\varepsilon(f_\varepsilon(T_\varepsilon(s,u_\varepsilon)) - f_\varepsilon(E_\varepsilon T_0(s,M_\varepsilon u_\varepsilon)))\|_{U_\varepsilon^p}\,ds$$

where we have used that $M_\varepsilon E_\varepsilon = I$ and that $f_\varepsilon(E_\varepsilon u) = E_\varepsilon f_0(u)$. Applying now Corollary 4.4 and Lemma 4.1 we have, for $0 < t \leqslant \tau$,

$$\|T_\varepsilon(t,u_\varepsilon) - E_\varepsilon T_0(t,M_\varepsilon u_\varepsilon)\|_{U_\varepsilon^p} \leqslant C e^{bt} t^{-\gamma}\rho(\varepsilon)\|u_\varepsilon\|_{U_\varepsilon^p} + C\rho(\varepsilon)\int_0^t (t-s)^{-\gamma} e^{b(t-s)}\|f_\varepsilon(T_\varepsilon(s,u_\varepsilon))\|_{U_\varepsilon^p}$$
$$+ C\int_0^t (t-s)^{\alpha-1}\|T_\varepsilon(s,u_\varepsilon) - E_\varepsilon T_0(s,M_\varepsilon u_\varepsilon)\|_{U_\varepsilon^p}$$

But since we have uniform bounds in $L^\infty(\Omega_\varepsilon)$ of all the attractors, the first two terms in the above inequality can be bounded by $C\rho(\varepsilon)t^{-\gamma}$. The result now follows applying the singular Gronwall's lemma (see [11]).

To show the uppersemicontinuity of the attractors $\mathcal{A}_\varepsilon$, we notice first that by the uniform $L^\infty(\Omega_\varepsilon)$ bounds of the attractors we have

$$\bigcup_{0\leqslant\varepsilon\leqslant\varepsilon_0} M_\varepsilon \mathcal{A}_\varepsilon$$

is a bounded set in U_0^∞. Hence, by the attractivity properties of $\mathcal{A}_0$, for a fixed $\eta > 0$ there exists a time $\tau > 0$ such that

$$\operatorname{dist}_{U_0^p}\Big(T_0(\tau)(M_\varepsilon\varphi_\varepsilon), \mathcal{A}_0\Big) \equiv \inf_{\varphi\in\mathcal{A}_0}\|T_0(\tau)(M_\varepsilon\varphi_\varepsilon) - \varphi\|_{U_0^p} \leqslant \eta, \quad \forall\varphi_\varepsilon\in\mathcal{A}_\varepsilon, \quad 0\leqslant\varepsilon\leqslant\varepsilon_0$$

which implies that

$$\operatorname{dist}_{U_\varepsilon^p}\Big(E_\varepsilon T_0(\tau)(M_\varepsilon\varphi_\varepsilon), E_\varepsilon\mathcal{A}_0\Big) \leqslant \eta, \quad \forall\varphi_\varepsilon\in\mathcal{A}_\varepsilon, \quad 0\leqslant\varepsilon\leqslant\varepsilon_0$$

Using the convergence of the nonlinear semigroups (5.3) with $t = \tau$, there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_1$,

$$\|T_\varepsilon(\tau,\varphi_\varepsilon) - E_\varepsilon T_0(\tau, M_\varepsilon\varphi_\varepsilon)\|_{U_\varepsilon^p} \leqslant \eta, \quad \forall\varphi_\varepsilon\in\mathcal{A}_\varepsilon, \quad 0\leqslant\varepsilon\leqslant\varepsilon_1.$$

Hence,

$$\operatorname{dist}_{U_\varepsilon^p}\Big(T_\varepsilon(\tau,\varphi_\varepsilon), E_\varepsilon\mathcal{A}_0\Big) \leqslant \eta, \quad \forall\varphi_\varepsilon\in\mathcal{A}_\varepsilon, \quad 0\leqslant\varepsilon\leqslant\varepsilon_1.$$

From the invariance $\mathcal{A}_\varepsilon$ we have that

$$\operatorname{dist}_{U_\varepsilon^p}\Big(\varphi_\varepsilon, E_\varepsilon\mathcal{A}_0\Big) \leqslant \eta, \quad \forall\varphi_\varepsilon\in\mathcal{A}_\varepsilon, \quad 0\leqslant\varepsilon\leqslant\varepsilon_1$$

which implies (5.4). ∎

Remark 5.2. *Observe that Proposition* 5.1 *proves the upper semicontinuity part of Theorem* 2.5*.*

6. Continuity of local unstable manifolds and of attractors

We already know that, if all equilibrium points of (2.7), which is the abstract version of (1.2), are hyperbolic then they are all isolated and there is only a finite number of them, say $\mathcal{E}_0 = \{e_0^1, \cdots, e_0^m\}$. In this case, we also know that there is an $\varepsilon_0 > 0$ such that the set of equilibria of (2.6), which is the abstract version of (1.1), $\mathcal{E}_\varepsilon = \{e_\varepsilon^1, \cdots, e_\varepsilon^m\}$ for all $0 < \varepsilon \leqslant \varepsilon_0$ and $e_\varepsilon^i \xrightarrow{E} e_0^i$ for $1 \leqslant i \leqslant m$ (see Theorem 2.3 of [3]). Moreover, we also know that the linear unstable manifolds associated to e_j^ε converge to the linear unstable manifold of e_j^ε, see Theorem 2.5 of [3]. For each $e_\varepsilon^j \in \mathcal{E}_\varepsilon$, $\varepsilon \in [0,1]$, we define its unstable manifold

$$W^u(e_\varepsilon^j) = \{\eta_\varepsilon \in U_\varepsilon^p : \text{ there is a global solution } \xi_\varepsilon : \mathbb{R} \to U_\varepsilon^p \text{ of } \\ (2.6) \text{ with } \xi_\varepsilon(0) = \eta_\varepsilon \text{ such that } \xi_\varepsilon(t) \xrightarrow{t\to-\infty} e_\varepsilon^j\}.$$

and its δ-local unstable manifold as

$$W_\delta^u(e_\varepsilon^j) = \{\eta_\varepsilon \in B(e_\varepsilon^j, \delta) \subset U_\varepsilon^p : \text{ there is a global solution } \xi_\varepsilon : \mathbb{R} \to U_\varepsilon^p \text{ of } \\ (2.6) \text{ with } \xi_\varepsilon(0) = \eta_\varepsilon, \ \xi_\varepsilon(t) \in B(e_\varepsilon^j, \delta), \forall\, t \leq 0, \text{ and } \xi_\varepsilon(t) \xrightarrow{t\to-\infty} e_\varepsilon^j\}.$$

These definitions are standard and we refer to [9] for further properties of local unstable manifolds.

In this section we show that the local unstable manifolds of e_ε^j, for $j = 1, \ldots, m$ fixed, behaves continuously with ε in U_ε^p.

Proposition 6.1. *Assume that $e_0 \in \mathcal{E}_0$ is hyperbolic; that is, $0 \notin \sigma(A_0 - f'(e_0)I)$. By Theorem 5.8 and Example 5.9 in* [3]*, there are $\delta > 0$ and ε_0 such that, there is a unique $e_\varepsilon \in \mathcal{E}_\varepsilon$ with $\|e_\varepsilon - E_\varepsilon e_0\|_{U_\varepsilon^p} < \delta$, for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Then, there is $\delta > 0$ such that*

$$\mathrm{dist}_{U_\varepsilon^p}(W_\delta^u(e_\varepsilon), E_\varepsilon W_\delta^u(e_0)) + \mathrm{dist}_{U_\varepsilon^p}(E_\varepsilon W_\delta^u(e_0), W_\delta^u(e_\varepsilon)) \xrightarrow{\varepsilon\to 0} 0$$

that is,

$$\sup_{u_\varepsilon \in W_\delta^u(e_\varepsilon)} \inf_{u_0 \in W_\delta^u(u_0)} \|u_\varepsilon - E_\varepsilon u_0\|_{U_\varepsilon^p} + \sup_{u_0 \in W_\delta^u(u_0)} \inf_{u_\varepsilon \in W_\delta^u(e_\varepsilon)} \|u_\varepsilon - E_\varepsilon u_0\|_{U_\varepsilon^p} \to 0, \ \textit{as } \varepsilon \to 0$$

Before proving this result, let us see how we can proceed to give a proof of our main result, Theorem 2.5.

Proof of Theorem 2.5: The upper-semicontinuity has already been proved in Proposition 5.1 from Section 5. Observe that to obtain the upper-semicontinuity of the attractors, we have used the continuity of the nonlinear semigroups, but no gradient structure of the flows have been used.

To obtain the lower-semicontinuity, we need to show that for each $\varphi_0 \in \mathcal{A}_0$ we have a sequence of $\varphi_\varepsilon \in \mathcal{A}_\varepsilon$, with the property that $\|\varphi_\varepsilon - E_\varepsilon \varphi_0\|_{U_\varepsilon^p} \to 0$ as $\varepsilon \to 0$. To accomplish this, we follow similar arguments as the one developed in [9], [10] or [2].

We are assuming that each equilibrium of the limiting problem $\mathcal{E}_0$ is hperbolic. This implies that we have a finite number of them and that the flow $T_0(t)$ has a gradient structure, see [4] and in particular, given $\varphi_0 \in \mathcal{A}_0$ it will lie in the unstable manifold of some $e_0 \in \mathcal{E}_0$. This implies that there exist an element $\phi_0 \in W_\delta^u(e_0)$ and a $\tau > 0$ such that $T_0(\tau, \phi_0) = \varphi_0$, where $\delta > 0$ is the one from Proposition 6.1. Using the continuity of the local unstable manifolds obtained in Proposition 6.1, we have that there exists a sequence of elements $\phi_\varepsilon \in W^u_\delta(e_\varepsilon)$ such that $\|\phi_\varepsilon - E_\varepsilon \phi_0\|_{U^p_\varepsilon} \to 0$. But, from the invariance of the attactor $\mathcal{A}_\varepsilon$ under the flow T_ε, we have $\varphi_\varepsilon = T_\varepsilon(\tau, \phi_\varepsilon) \in \mathcal{A}_\varepsilon$. Moreover,

$$\begin{aligned}\|\varphi_\varepsilon - E_\varepsilon\varphi_0\|_{U^p_\varepsilon} &= \|T_\varepsilon(\tau,\phi_\varepsilon) - E_\varepsilon T_0(\tau,\phi_0)\|_{U^p_\varepsilon}\\ &\leqslant \|T_\varepsilon(\tau,\phi_\varepsilon) - E_\varepsilon T_0(\tau, M_\varepsilon\phi_\varepsilon)\|_{U^p_\varepsilon} + \|E_\varepsilon T_0(\tau, M_\varepsilon\phi_\varepsilon) - E_\varepsilon T_0(\tau,\phi_0)\|_{U^p_\varepsilon}\\ &\leqslant M(\tau)\tau^{-\gamma}c(\varepsilon) + \|T_0(\tau, M_\varepsilon\phi_\varepsilon) - T_0(\tau,\phi_0)\|_{U^p_0}\end{aligned}$$

where we are using (5.3) and the fact that $\|E_\varepsilon\|_{\mathcal{L}(U^p_0, U^p_\varepsilon)} = 1$.

The continuity of the map $T(\tau,\cdot) : U^p_0 \to U^p_0$, the fact that $\|\phi_0 - M_\varepsilon\phi_\varepsilon\|_{U^p_0} \to 0$ as $\varepsilon \to 0$ and that $c(\varepsilon) \to 0$, shows that $\|\varphi_\varepsilon - E_\varepsilon\varphi_0\|_{U^p_\varepsilon} \to 0$ as $\varepsilon \to 0$. This concludes the proof of Theorem 2.5. ∎

Proof of Proposition 6.1: Let $\{e_\varepsilon\}$ with $e_\varepsilon \in \mathcal{E}_\varepsilon$, $\varepsilon \in [0,1]$, such $\|e_\varepsilon - E_\varepsilon e_0\|_{U^p_\varepsilon} \overset{\varepsilon\to 0}{\to} 0$. Rewriting (2.6) for $w_\varepsilon = u_\varepsilon - e_\varepsilon$ to deal with the neighborhood of e_ε we arrive at

$$w_t + A_\varepsilon w - f'_\varepsilon(e_\varepsilon)w = f(w + e_\varepsilon) - f(e_\varepsilon) - f'_\varepsilon(e_\varepsilon)w. \tag{6.1}$$

Let us denote by $V_0 = -f'(e_0)$, $V_\varepsilon = -f'(e_\varepsilon)$. Using the hiperbolicity of e_0, e_ε we consider $b < 0$ and define σ^+_ε, $Q(\sigma^*_\varepsilon)$ as in (4.10), see Remark 4.5.

Decomposing (6.1) with the aid of projection $Q(\sigma^+_\varepsilon)$ and denoting by $\tilde{A}_\varepsilon$ the restriction of $A_\varepsilon + V_\varepsilon$ to the kernel of $Q(\sigma^+_\varepsilon)$, by B_ε the restriction of $A_\varepsilon + V_\varepsilon$ to the range of $Q(\sigma^+_\varepsilon)$ and making $S^{-1}_\varepsilon v = Q(\sigma^+_\varepsilon)w$, $z = (I - Q(\sigma^+_\varepsilon))w$ we rewrite (6.1) as

$$\begin{aligned}&\dot{v} + B_\varepsilon v = Q(\sigma^+_\varepsilon)F_\varepsilon(S_\varepsilon v, z)\\ &\dot{z} + \tilde{A}_\varepsilon z = (I - Q(\sigma^+_\varepsilon))F_\varepsilon(S_\varepsilon v, z),\end{aligned} \tag{6.2}$$

where $F_\varepsilon(0,0) = 0$ and $F'_\varepsilon(0,0) = 0$. Proceeding as in Example 5.9 in [3] we have that, given $\rho > 0$ there is a $\delta > 0$ such that

$$\begin{aligned}&\|F_\varepsilon(S_\varepsilon v, z)\|_{U^q_\varepsilon} < \rho,\\ &\|F_\varepsilon(S_\varepsilon v, z) - F_\varepsilon(S_\varepsilon\tilde{v}, \tilde{z})\|_{U^q_\varepsilon} < \rho(\|v - \tilde{v}\|_{\mathbb{R}^n} + \|z - \tilde{z}\|_{U^p_\varepsilon}).\end{aligned} \tag{6.3}$$

for all $(v, z) \in B_\delta(0,0)$ and for all $\varepsilon \in (0,1]$. Since we are interested only in the behavior of the solutions near $(0,0)$ we cut F_ε outside $B_\delta(0,0)$ in such a way that it satisfies (6.3) globally.

Proceeding as in [2, 7] we can show that for a suitably small $\rho > 0$, there is an unstable manifold for e_ε

$$S^\varepsilon = \{(v,z) : z = \Sigma^*_\varepsilon(v),\ v \in \mathbb{R}^n\}$$

where $\Sigma^*_\varepsilon : \mathbb{R}^n \to \mathrm{Ker}(Q_\varepsilon)$ is bounded and Lipschitz continuous. Furthermore

$$\sup_{v\in\mathbb{R}^n} \|\Sigma^*_\varepsilon(v) - E_\varepsilon\Sigma^*_0(v)\|_{U^p_\varepsilon} \overset{\varepsilon\to 0}{\to} 0.$$

Let us sketch the proof of existence of the unstable manifold as a graph and prove its continuity. Let $\Sigma_\varepsilon : \mathbb{R}^n \to \mathrm{Ker}(Q_\varepsilon)$ such that

$$|\!|\!|\Sigma_\varepsilon|\!|\!| := \sup_{v\in\mathbb{R}^n} \|\Sigma_\varepsilon(v)\|_{U^p_\varepsilon} \leqslant D,\ \ \|\Sigma_\varepsilon(v) - \Sigma_\varepsilon(\tilde{v})\|_{U^p_\varepsilon} \leqslant L\|v - \tilde{v}\|_{\mathbb{R}^n}. \tag{6.4}$$

If $v_\varepsilon(t)=\psi(t,\tau,\eta,\Sigma_\varepsilon)$ denotes the solution of

$$\frac{dv_\varepsilon}{dt}+B_\varepsilon v_\varepsilon=F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma_\varepsilon(v_\varepsilon)),\quad \text{for } t<\tau,\ v_\varepsilon(\tau)=\eta,$$

We seek for a fixed point Σ_ε^* of

$$\Phi(\Sigma_\varepsilon)(\eta)=\int_{-\infty}^{\tau}e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma_\varepsilon^+))F_\varepsilon(S_\varepsilon v_\varepsilon(s),\Sigma_\varepsilon(v_\varepsilon(s)))ds,\quad \varepsilon\in[0,1]. \tag{6.5}$$

in the class of Lipschitz maps $\Sigma_\varepsilon:\mathbb{R}^n\to \mathrm{Ker}(Q_\varepsilon)$ which are globally bounded with bound D and globally Lipschitz with Lipschitz constant L.

Note that, from (4.13),

$$\|\Phi(\Sigma_\varepsilon)(\eta)\|_{U_\varepsilon^p}=\int_{-\infty}^{\tau}\rho C(\tau-s)^{-\gamma}e^{-b(\tau-s)}ds. \tag{6.6}$$

and for suitably chosen ρ we have that $|\!|\!|\Phi(\Sigma_\varepsilon)|\!|\!|\leqslant D$.

Next, suppose that Σ_ε and $\tilde{\Sigma}_\varepsilon$ are functions satisfying (6.4), η, $\tilde{\eta}\in\mathbb{R}^n$ and denote $v_\varepsilon(t)=\psi(t,\tau,\eta,\Sigma_\varepsilon)$, $\tilde{v}_\varepsilon(t)=\psi(t,\tau,\tilde{\eta},\tilde{\Sigma}_\varepsilon)$. Then,

$$v_\varepsilon(t)-\tilde{v}_\varepsilon(t)=e^{-B_\varepsilon(t-\tau)}(\eta-\tilde{\eta})+\int_\tau^t e^{-B_\varepsilon(t-s)}Q_\varepsilon[F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma_\varepsilon(v_\varepsilon))-F_\varepsilon(S_\varepsilon\tilde{v}_\varepsilon,\tilde{\Sigma}_\varepsilon(\tilde{v}_\varepsilon))]ds.$$

And

$$\begin{aligned}
\|v_\varepsilon(t)-\tilde{v}_\varepsilon(t)\|_{\mathbb{R}^n}\leqslant{}& Ce^{b(t-\tau)}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}\\
&+C\int_t^\tau e^{b(t-s)}\|Q_\varepsilon F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma_\varepsilon(v_\varepsilon))-Q_\varepsilon F_\varepsilon(S_\varepsilon\tilde{v}_\varepsilon,\tilde{\Sigma}_\varepsilon(\tilde{v}_\varepsilon))\|_{\mathbb{R}^n}ds\\
\leqslant{}& Ce^{b(t-\tau)}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}\\
&+\rho C\int_t^\tau e^{-b(t-s)}\left(\|\Sigma_\varepsilon(v_\varepsilon)-\tilde{\Sigma}_\varepsilon(\tilde{v}_\varepsilon)\|_{U_\varepsilon^p}+\|v_\varepsilon-\tilde{v}_\varepsilon\|_{\mathbb{R}^n}\right)ds\\
\leqslant{}& Ce^{b(t-\tau)}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}\\
&+\rho C\int_t^\tau e^{b(t-s)}\left(\|\Sigma_\varepsilon(\tilde{v}_\varepsilon)-\tilde{\Sigma}_\varepsilon(\tilde{v}_\varepsilon)\|_{U_\varepsilon^p}+(1+L)\|v_\varepsilon-\tilde{v}_\varepsilon\|_{\mathbb{R}^n}\right)ds\\
\leqslant{}& Ce^{b(t-\tau)}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}\\
&+\rho C\int_t^\tau e^{b(t-s)}\left((1+L)\|v_\varepsilon-\tilde{v}_\varepsilon\|_{\mathbb{R}^n}+|\!|\!|\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|\!|\!|_{U_\varepsilon^p}\right)ds\\
\leqslant{}& Ce^{b(t-\tau)}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}\\
&+\rho C(1+L)\int_t^\tau e^{b(t-s)}\|v_\varepsilon-\tilde{v}_\varepsilon\|_{\mathbb{R}^n}ds+\rho C|\!|\!|\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|\!|\!|_{U_\varepsilon^p}\int_t^\tau e^{b(t-s)}ds.
\end{aligned}$$

Let $\phi(t)=e^{-b(t-\tau)}\|v_\varepsilon(t)-\tilde{v}_\varepsilon(t)\|_{\mathbb{R}^n}$. Then,

$$\phi(t)\leqslant C\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}+\rho C\int_t^\tau e^{b(\tau-s)}ds|\!|\!|\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|\!|\!|_{U_\varepsilon^p}+C\,\rho\,(1+L)\int_t^\tau\phi(s)ds.$$

By Gronwall's inequality

$$\begin{aligned}
\|v_\varepsilon(t)-\tilde{v}_\varepsilon(t)\|_{\mathbb{R}^n}&\leqslant[C\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}e^{b(t-\tau)}+\rho C\int_t^\tau e^{b(t-s)}ds|\!|\!|\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|\!|\!|_{U_\varepsilon^p}]e^{-\rho C(1+L)(t-\tau)}\\
&\leqslant[C\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}+\rho Cb^{-1}|\!|\!|\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|\!|\!|_{U_\varepsilon^p}]e^{-\rho C(1+L)(t-\tau)}
\end{aligned}$$

Thus,

$$\begin{aligned}\|\Phi(\Sigma_\varepsilon)(\eta)- &\Phi(\tilde{\Sigma}_\varepsilon)(\tilde{\eta})\|_{U^p_\varepsilon}\\ &\leqslant C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\|F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma_\varepsilon(v_\varepsilon))-F_\varepsilon(S_\varepsilon\tilde{v}_\varepsilon,\tilde{\Sigma}_\varepsilon(\tilde{v_\varepsilon}))\|_{L^2(\Omega_\varepsilon)}ds\\ &\leqslant \rho C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\left(\|\Sigma_\varepsilon(v_\varepsilon)-\tilde{\Sigma}_\varepsilon(\tilde{v_\varepsilon})\|_{U^p_\varepsilon}+\|v_\varepsilon-\tilde{v_\varepsilon}\|_{\mathbb{R}^n}\right)ds\\ &\leqslant \rho C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\left[(1+L)\|v_\varepsilon-\tilde{v_\varepsilon}\|_{\mathbb{R}^n}+|||\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|||\right]ds.\end{aligned}$$

Using the estimates for $\|v_\varepsilon-\tilde{v_\varepsilon}\|_{\mathbb{R}^n}$ we obtain

$$\begin{aligned}\|\Phi(\Sigma_\varepsilon)(\eta)-\Phi(\tilde{\Sigma}_\varepsilon)(\tilde{\eta})\|\leqslant \rho C\Gamma(1-\gamma)\left[b^{-1+\gamma}+\frac{\rho C(1+L)}{b(b-\rho C(1+L))^{1-\gamma}}\right]|||\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|||\\ +\frac{\rho C^2(1+L)\Gamma(1-\gamma)}{(b-\rho C(1+L))^{-1+\gamma}}\|\eta-\tilde{\eta}\|_{\mathbb{R}^n}.\end{aligned}$$

Let

$$I_\Sigma(\rho)=\rho C\Gamma(1-\gamma)\left[b^{-1+\gamma}+\frac{\rho C(1+L)}{b(b-\rho C(1+L))^{1-\gamma}}\right]$$

and

$$I_\eta(\rho)=\frac{\rho C^2(1+L)\Gamma(1-\gamma)}{(b-\rho C(1+L))^{1-\gamma}}.$$

It is easy to see that, given $\theta<1$, there exists a ρ_0 such that, for $\rho\leqslant\rho_0$, $I_\Sigma(\rho)\leqslant\theta$ and $I_\eta(\rho)\leqslant L$ and

$$\|\Phi(\Sigma_\varepsilon)(\eta)-\Phi(\tilde{\Sigma}_\varepsilon)(\tilde{\eta})\|_{U^p_\varepsilon}\leqslant L\|\eta-\eta'\|_{\mathbb{R}^n}+\theta|||\Sigma_\varepsilon-\tilde{\Sigma}_\varepsilon|||. \tag{6.7}$$

The inequalities (6.6) and (6.7) imply that G is a contraction map from the class of functions that satisfy (6.4) into itself. Therefore, it has a unique fixed point $\Sigma^*_\varepsilon=\Phi(\Sigma^*_\varepsilon)$ in this class. The invariance follows in the usual manner.

The fact that the graph is the whole unstable manifold follows (taking the limit as t_0 tends to $-\infty$) from the following: If $w(t)=(v(t),z(t))$, $t\in\mathbb{R}$, is a global solution of (6.1) which is bounded as $t\to-\infty$, there are constants $\tilde{M}\geqslant1$ and $\nu>0$ such that

$$\|z(t)-\Sigma^*_\varepsilon(v(t))\|_{U^p_\varepsilon}\leqslant\tilde{M}(t-t_0)^{-\gamma}e^{-\nu(t-t_0)}\|z(t_0)-\Sigma^*_\varepsilon(v(t_0))\|_{U^p_\varepsilon},\quad t_0<t. \tag{6.8}$$

The proof of (6.8) can be carried out following the steps in the proof of (A.8) in [6], using the singular Gronwall's inequality instead of the usual one, and noting that ε can be considered fixed for this purpose.

It remains to prove the continuity of the unstable manifolds. This is accomplished in the following manner. If $0\leqslant\varepsilon\leqslant\varepsilon_0$ is such that the unstable manifold is given by the graph of Σ^*_ε, $0\leqslant\varepsilon\leqslant\varepsilon_0$, we want to show that

$$\sup_{\eta\in\mathbb{R}^n}\|\Sigma^*_\varepsilon(\eta)-E_\varepsilon\Sigma^*_0(\eta)\|_{U^p_\varepsilon}=|||\Sigma^*_\varepsilon-E_\varepsilon\Sigma^*_0|||.$$

It follows from Proposition 4.3 that

$$
\begin{aligned}
&\|\Sigma^*_\varepsilon(\eta_\varepsilon) - E_\varepsilon\Sigma^*_0(\eta)\|_{U^p_\varepsilon}\\
&\leqslant \int_{-\infty}^{\tau} \|e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma^+_\varepsilon))F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon)) - E_\varepsilon e^{-\tilde{A}_0(\tau-s)}(I-Q(\sigma^+_0))F_0(S_0v_0,\Sigma^*_0(v_0))\|_{U^p_\varepsilon}ds\\
&\leqslant \int_{-\infty}^{\tau} \|e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma^+_\varepsilon))[F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon)) - E_\varepsilon F_0(S_0v_0,\Sigma^*_0(v_0))]\|_{U^p_\varepsilon}ds\\
&+ \int_{-\infty}^{\tau} \|[e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma^+_\varepsilon) - E_\varepsilon e^{-\tilde{A}_0(\tau-s)}(I-Q(\sigma^+_0)M_\varepsilon]E_\varepsilon F_0(S_0v_0,\Sigma^*_0(v_0))\|_{U^p_\varepsilon}ds\\
&\leqslant \int_{-\infty}^{\tau} \|e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma^+_\varepsilon))[F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon)) - F_\varepsilon(E_\varepsilon(S_\varepsilon v_0,\Sigma^*_0(v_0)))]\|_{U^p_\varepsilon}ds\\
&+ \int_{-\infty}^{\tau} \|[e^{-\tilde{A}_\varepsilon(\tau-s)}(I-Q(\sigma^+_\varepsilon) - E_\varepsilon e^{\tilde{A}_0(\tau-s)}(I-Q(\sigma^+_0)M_\varepsilon]E_\varepsilon F_0(S_0v_0,\Sigma^*_0(v_0))\|_{U^p_\varepsilon}ds\\
&\leqslant C\int_{-\infty}^{\tau} e^{b(\tau-s)}(\tau-s)^{-\gamma}\|F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon)) - F_\varepsilon(E_\varepsilon(S_0v_0,\Sigma^*_0(v_0)))\|_{U^q_\varepsilon}ds\\
&+ C\rho(\varepsilon)\int_{-\infty}^{\tau} e^{b(\tau-s)}\|F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon))\|_{C(\bar{\Omega}_\varepsilon)}ds\\
&\leqslant \rho\, Cb^{-1}\rho(\varepsilon) + \rho Cb^{\gamma-1}\Gamma(1-\gamma)\,|||\Sigma^*_\varepsilon - E_\varepsilon\Sigma^*_0|||\\
&+ \rho\, C(1+L)\int_{-\infty}^{\tau} e^{-b(\tau-s)}(\tau-s)^{-\gamma}\|v_\varepsilon - v_0\|_{\mathbb{R}^n}ds.
\end{aligned}
\tag{6.9}
$$

Thus, it is enough to estimate $\|v_\varepsilon - v_0\|_{\mathbb{R}^n}$. Note that

$$
\begin{aligned}
\|v_\varepsilon - v_0\|_{\mathbb{R}^n} &\leqslant \int_t^{\tau} \|e^{-B_\varepsilon(t-s)} - e^{-B_0(t-s)}\|\,\|F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon))\|_{\mathbb{R}^n}ds\\
&+\int_t^{\tau} \|e^{-B_0(t-s)}\|\,\|F_\varepsilon(S_\varepsilon v_\varepsilon,\Sigma^*_\varepsilon(v_\varepsilon)) - F_0(S_0v_0,\Sigma^*_0(v_0))\|_{\mathbb{R}^n}ds\\
&\leqslant \rho Mb^{-1}[o(1) + |||\Sigma^*_\varepsilon - \Sigma^*_0|||] + \rho C(1+L)\int_t^{\tau} e^{b(t-s)}\|v_\varepsilon - v_0\|_{\mathbb{R}^n}ds
\end{aligned}
$$

Therefore

$$\|v_\varepsilon - v_0\|_{\mathbb{R}^n} \leqslant \rho Cb^{-1}\left[o(1) + |||\Sigma^*_\varepsilon - \Sigma^*_0|||\right] e^{-\rho C(1+L)(\tau-t)}$$

which shows that

$$\sup_{\eta\in\mathbb{R}^n} \|\Sigma^*_\varepsilon(\eta) - \Sigma^*_0(\eta)\|_{U^p_\varepsilon} \overset{\varepsilon\to 0}{\to} 0.$$

This proves the result. ■

7. Continuity of Attractors in Other Norms

In this section we study the continuity of attractors in other norms and very specially in the norm of the space $U^{1,2}_\varepsilon$, see (2.8). This continuity is obtained as a consequence of the regularization properties of the nonlinear semigroups. As a matter of fact, in many instances the attractors $\mathcal{A}_\varepsilon$, $\mathcal{A}_0$ live in better spaces X_ε and X_0 respectively for which the linear map

$E_\varepsilon : X_0 \to X_\varepsilon$ is well defined as well. We would like to give conditions that, once the continuity of the attractors in U^p_ε is obtained, will guarantee the continuity results for the attractors in these better spaces. In fact, the following result holds

Proposition 7.1. *If there exists a $\tau > 0$ fixed such that for each sequence of $\varepsilon_n \to 0$, $\phi_{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}$ and $\phi_0 \in \mathcal{A}_0$ with $\|\phi_{\varepsilon_n} - E_{\varepsilon_n}\phi_0\|_{U^p_{\varepsilon_n}} \to 0$ implies that*

$$\|T_{\varepsilon_n}(\tau, \phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, \phi_0)\|_{X_{\varepsilon_n}} \to 0 \tag{7.1}$$

then, the upper semicontinuity of the attractors in U^p_ε implies the upper semicontinuity in X_ε and the lower semicontinuity of the attractors in U^p_ε implies the lower semicontinuity of the attractors in X_ε.

Proof: Assume we have a family of $\varphi_\varepsilon \in \mathcal{A}_\varepsilon$. From the invariance of the attractors under the semigroup T_ε, we have that there exist $\phi_\varepsilon \in \mathcal{A}_\varepsilon$ with $T_\varepsilon(\tau, \phi_\varepsilon) = \varphi_\varepsilon$.

If the attractors are E_ε-upper semicontinuous in U^p_ε, we have that for each sequence $\varepsilon_n \to 0$, there will exist a subsequence, that we still denote by ε_n and an element $\phi_0 \in \mathcal{A}_0$ such that $\|\phi_{\varepsilon_n} - E_{\varepsilon_n}\phi_0\|_{U^p_{\varepsilon_n}} \to 0$ as $\varepsilon_n \to 0$. With (7.1) we get that if we define $\varphi_0 = T_0(\tau, \phi_0)$, we have $\|\varphi_{\varepsilon_n} - E_{\varepsilon n}\varphi_0\|_{X_{\varepsilon_n}} \to 0$, which shows the E_ε-upper semicontinuity in X_ε.

Assume now that the attractors are E_ε-lower semicontinuous in U^p_ε. If $\varphi_0 \in \mathcal{A}_0$ and if we define $\phi_0 \in \mathcal{A}_0$ with $T_0(\tau, \phi_0) = \varphi_0$, then there will exist a sequence of $\phi_\varepsilon \in \mathcal{A}_\varepsilon$ with $\|\phi_\varepsilon - E_\varepsilon\phi_0\|_{U^p_\varepsilon} \to 0$ as $\varepsilon \to 0$. Using (7.1) again, we get that $\|\varphi_\varepsilon - E_\varepsilon\varphi_0\|_{X_\varepsilon} \to 0$ which shows the E_ε-lower semicontinuty in X_ε. ∎

With this result we can provide now a proof of Theorem 2.7.

Proof of Theorem 2.7: We will apply Proposition 7.1, proving first that

$$\|T_{\varepsilon_n}(\tau, \phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\varepsilon_n}} \to 0$$

for some $\tau > 0$ fixed, sequences $\varepsilon_n \to 0$, $\phi_{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}$ and $\phi_0 \in \mathcal{A}_0$ with $\|\phi_{\varepsilon_n} - E_{\varepsilon_n}\phi_0\|_{U^p_{\varepsilon_n}} \to 0$.

Observe first that

$$\begin{aligned} &\|T_{\varepsilon_n}(\tau, \phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\varepsilon_n}} \leqslant \\ &\quad \|T_{\varepsilon_n}(\tau, \phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, M_\varepsilon\phi_{\varepsilon_n})\|_{U^{1,2}_{\varepsilon_n}} + \|E_\varepsilon T_0(\tau, M_\varepsilon\phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\varepsilon_n}} \end{aligned} \tag{7.2}$$

and for a fixed $\tau > 0$,

$$\|E_\varepsilon T_0(\tau, M_\varepsilon\phi_{\varepsilon_n}) - E_{\varepsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\varepsilon_n}} \leqslant \|T_0(\tau, M_\varepsilon\phi_{\varepsilon_n}) - T_0(\tau, \phi_0)\|_{U^{1,2}_0} \to 0$$

since $T_0(\tau, \cdot) : U^p_0 \to U^{1,2}_0$ is continuous, see [4] .

To estimate the first term of the second line of (7.2) we use the Variation of Constants Formula (5.1) for $\varepsilon \in [0, 1]$ and with simple computations we obtain

$$\begin{aligned} \|T_\varepsilon(t,\phi_\varepsilon) - E_\varepsilon T_0(t, M_\varepsilon\phi_\varepsilon)\|_{U^{1,2}_\varepsilon} &\leqslant \|e^{-A_\varepsilon t}\phi_\varepsilon - E_\varepsilon e^{-A_0 t}M_\varepsilon\phi_\varepsilon\|_{U^{1,2}_\varepsilon} \\ &+ \int_0^t \|\left(e^{-A_\varepsilon(t-s)} - E_\varepsilon e^{-A_0(t-s)}M_\varepsilon\right) f_\varepsilon(T_\varepsilon(s, \phi_\varepsilon))\|_{U^{1,2}_\varepsilon} \, ds \\ &+ \int_0^t \|E_\varepsilon e^{-A_0(t-s)}M_\varepsilon(f_\varepsilon(T_\varepsilon(s, \phi_\varepsilon)) - f_\varepsilon(E_\varepsilon T_0(s, M_\varepsilon\phi_\varepsilon)))\|_{U^{1,2}_\varepsilon} \, ds, \quad \varepsilon \in [0, \varepsilon_0]. \end{aligned} \tag{7.3}$$

But note that $\mathcal{A}_\varepsilon \subset C(\bar\Omega_\varepsilon)$ for $0<\varepsilon\leqslant\varepsilon_0$, $\mathcal{A}_0\subset C(\bar\Omega)\oplus C([0,1])$ and that we have uniform bounds in these spaces.

If we are able to obtain the following two estimates:

$$\|e^{-A_\varepsilon t}-E_\varepsilon e^{-A_0 t}M_\varepsilon\|_{\mathcal{L}(C(\bar\Omega)\oplus C(\bar R_\varepsilon),U^{1,2}_\varepsilon)}\leqslant Ct^{-\gamma}\nu(\varepsilon),\ t>0. \tag{7.4}$$

for some $0\leqslant\gamma<1$ and with $\nu(\varepsilon)\to 0$ as $\varepsilon\to 0$, and

$$\|e^{-A_0 t}\|_{\mathcal{L}(U^p_0,U^{1,2}_0)}\leqslant Ct^{-\beta},\quad t>0. \tag{7.5}$$

for some $0\leqslant\beta<1$, then using (7.4) and (7.5) in (7.3) and using the convergence of the nonlinear semigroup in U^p_0 we obtain that $\|T_\varepsilon(t,\phi_\varepsilon)-E_\varepsilon T_0(t,M_\varepsilon\phi_\varepsilon)\|_{U^{1,2}_\varepsilon}\to 0$ as $\varepsilon\to 0$.

The proof of (7.5) is in [4] Remark 3.2.

Hence we just need to show (7.4). To obtain this estimate we need some extra resolvent estimates, similar to the ones obtained in Section 3.1. To that end we introduce the continuous extension operator

$$\begin{array}{rccl} E^{\mathcal{C}}_\varepsilon: & \mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(0,1) & \to & \mathcal{C}(\overline{\Omega_\varepsilon}) \\ & (w_\varepsilon,v_\varepsilon) & \to & E^{\mathcal{C}}_\varepsilon(w_\varepsilon,v_\varepsilon) = \begin{cases} w_\varepsilon, & x\in\Omega \\ \tilde v_\varepsilon, & x\in \mathrm{R}_\varepsilon, \end{cases}\end{array} \tag{7.6}$$

where

$$\tilde v_\varepsilon(x)=v_\varepsilon(s)+h_\varepsilon(s)\left(w_\varepsilon(0,y)-v_\varepsilon(0)\right)+h_\varepsilon(1-s)\left(w_\varepsilon(1,y)-v_\varepsilon(1)\right),\ x=(s,y)\in\mathrm{R}_\varepsilon, \tag{7.7}$$

and the function $h_\delta(s)=h(\frac{s}{\delta})$, where $h:\mathbb{R}^+\to[0,1]$ is a $\mathcal{C}^\infty$ function such that

$$h(s)=\begin{cases}1, & \text{for } s\in[0,1/4],\\ 0, & \text{for } s\geqslant 3/4\end{cases}$$

and $|h'(s)|\leqslant C$.

Observe that with this definition $E^{\mathcal{C}}_\varepsilon(w_\varepsilon,v_\varepsilon)$ is always a continuous function in $\bar\Omega_\varepsilon$ if $(w_\varepsilon,v_\varepsilon)\in\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(0,1)$. Moreover, if $(w_\varepsilon,v_\varepsilon)\in U^{1,2}_0$ then, $E^{\mathcal{C}}_\varepsilon(w_\varepsilon,v_\varepsilon)\in H^1(\Omega_\varepsilon)$.

We also need the following lemmas whose proofs will be provided later.

Lemma 7.2. *Let $\lambda\in\rho(A_\varepsilon)\cap\rho(A_0)$, then the following holds*

$$\begin{aligned}(\lambda+A_\varepsilon)^{-1}-E_\varepsilon(\lambda+A_0)^{-1}M &= (I-\lambda(A_\varepsilon+\lambda)^{-1})(A_\varepsilon^{-1}-E^{\mathcal{C}}_\varepsilon A_0^{-1}M_\varepsilon)(I-\lambda E_\varepsilon(A_0+\lambda)^{-1}M_\varepsilon)\\ &\quad + (I-\lambda(A_\varepsilon+\lambda)^{-1})\left(E^{\mathcal{C}}_\varepsilon-E_\varepsilon\right)(A_0+\lambda)^{-1}M_\varepsilon\end{aligned}$$

Lemma 7.3. *There is a constant $C>0$ such that for each $\lambda\in\Sigma_\theta$ we have*

$$\begin{aligned}\|(E^{\mathcal{C}}_\varepsilon-E_\varepsilon)(A_0+\lambda)^{-1}M_\varepsilon\|_{\mathcal{L}(\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(\bar R_\varepsilon),H^1(\Omega)\oplus H^1(R_\varepsilon))} &\leqslant C\frac{\varepsilon^{\frac{N}{2}}}{1+|\lambda|^{\frac12}},\\ \|(I-\lambda(A_\varepsilon+\lambda)^{-1})(E^{\mathcal{C}}_\varepsilon-E_\varepsilon)(A_0+\lambda)^{-1}M_\varepsilon\|_{\mathcal{L}(\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(\bar R_\varepsilon),H^1(\Omega)\oplus H^1(R_\varepsilon))} &\leqslant C\varepsilon^{N/2}.\end{aligned} \tag{7.8}$$

Lemma 7.4. *There is a constant $C>0$, independent of ε, such that*

(i) $\|E_\varepsilon(I-\lambda(A_0+\lambda)^{-1})M_\varepsilon f_\varepsilon\|_{C(\bar\Omega)\oplus C(\bar R_\varepsilon)}\leqslant C\,\|f_\varepsilon\|_{C(\bar\Omega)\oplus C(\bar R_\varepsilon)}$,

(ii) $\|(I-\lambda(A_\varepsilon+\lambda)^{-1})g_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqslant C\,\|g_\varepsilon\|_{H^1(\Omega_\varepsilon)}$.

Lemma 7.5. *There exists a constant $C > 0$ such that for all $\lambda \in \Sigma_\theta$ and all $f_\varepsilon \in \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(R_\varepsilon)$,*

$$\left\|((A_\varepsilon + \lambda)^{-1} - E_\varepsilon(A_0 + \lambda)^{-1} M_\varepsilon) f_\varepsilon\right\|_{H^1(\Omega)\oplus H^1(R_\varepsilon)} \leqslant C\varepsilon^{N/2} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(R_\varepsilon)}. \tag{7.9}$$

Clearly, from Lemma 7.5 and the expression of the differences of the semigroups in terms of the integral of the difference of the resolvents as in (4.9), we have that there is a constant $C > 0$ such that

$$\left\|e^{-A_\varepsilon t} - E_\varepsilon e^{-A_0 t} M_\varepsilon\right\|_{\mathcal{L}(C(\bar{\Omega})\oplus C(\bar{R}_\varepsilon), H^1(\Omega)\oplus H^1(R_\varepsilon))} \leqslant C\varepsilon^{N/2} t^{-1}. \tag{7.10}$$

On the other hand,

$$\begin{aligned}
&\left\|e^{-A_\varepsilon t} - E_\varepsilon e^{-A_0 t} M_\varepsilon\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega)\oplus H^1(R_\varepsilon))} \leqslant \\
&\quad \left\|e^{-A_\varepsilon t}\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega)\oplus H^1(R_\varepsilon))} + \left\|E_\varepsilon e^{-A_0 t} M_\varepsilon\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega)\oplus H^1(R_\varepsilon))} \leqslant \\
&\quad \left\|e^{-A_\varepsilon t}\right\|_{\mathcal{L}(L^2(\Omega_\varepsilon), H^1(\Omega_\varepsilon))} + \left\|e^{-A_0 t}\right\|_{\mathcal{L}(U_0^p, H^1(\Omega)\oplus H^1(0,1))} \leqslant C t^{-\beta}
\end{aligned} \tag{7.11}$$

for some β with $1/2 < \beta < 1$, see [4], Remark 3.2. . Interpolating (7.10) and (7.11), we have that that, for any $\eta < 1$,

$$\left\|e^{-A_\varepsilon t} - E_\varepsilon e^{-A_0 t} M_\varepsilon\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega)\oplus H^1(R_\varepsilon))} \leqslant C\varepsilon^{\eta N/2} t^{-(\eta + (1-\eta)\beta)}. \tag{7.12}$$

Choosing $\frac{N-1}{N} < \eta < 1$ so that $\eta N/2 > (N-1)/2$, the result follows with $\gamma = \eta + (1-\eta)\beta < 1$. This shows estimate (7.4) and the theorem is proved. ∎

Remark 7.6. *We may also obtain the convergence of the attractors in some other norms. As a matter of fact if K is a compact subset of $\bar{\Omega}\backslash\{P_0, P_1\}$ we can easily obtain uniform bounds of all the attractors for instance in $C^{1,\eta}(K)$. This estimates may be obtained with an appropriate cut-off function and using standard regularity properties of the nonlinear semigroups (we are far away from the channel R_ε). Hence, since we have obtained already the continuity (lower or upper) of the attractor in $L^p(K)$, with the compact embedding of $C^{1,\eta}(K)$ in $C^{1,\eta^-}(K)$ we also get the continuity (lower or upper) in $C^{1,\eta^-}(K)$.*

We provide now the proofs of the different lemmas we have stated above.

Proof of Lemma 7.2: This lemma is obtained in a similar way as Lemma 3.5. ∎

Proof of Lemma 7.3: Let $f_\varepsilon \in \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(\bar{R}_\varepsilon)$ and define $K_\varepsilon := (E_\varepsilon^{\mathcal{C}} - E_\varepsilon)(A_0 + \lambda)^{-1} M f_\varepsilon = \tilde{z}_\varepsilon - z_\varepsilon$,, where $\tilde{z}_\varepsilon = E_{\mathcal{C}}(A_0 + \lambda)^{-1} M f_\varepsilon$ e $z_\varepsilon = E_\varepsilon(A_0 + \lambda)^{-1} M f_\varepsilon$.

Observe that $(A_0 + \lambda)^{-1} M f_\varepsilon = (w_\varepsilon, v_\varepsilon)$ where

$$\begin{cases}
-\Delta w_\varepsilon + \lambda w_\varepsilon = f_\varepsilon, & x \in \Omega \\
\frac{\partial w_\varepsilon}{\partial n} = 0, & x \in \partial\Omega \\
-\dfrac{1}{g}(g v_{\varepsilon s})_s + \lambda v_\varepsilon = M f_\varepsilon, & s \in (0,1) \\
v_\varepsilon(0) = w_\varepsilon(0), v_\varepsilon(1) = w_\varepsilon(1), &
\end{cases} \tag{7.13}$$

$\tilde{v}_\varepsilon(s,y) = v_\varepsilon(s) + h_\varepsilon(s)\,(w_\varepsilon(0,y) - v_\varepsilon(0)) + h_\varepsilon(1-s)\,(w_\varepsilon(1,y) - v_\varepsilon(1))$, $\forall (s,y) \in R_\varepsilon$ and $z_\varepsilon(s,y) = v_\varepsilon(s)$, $\forall (s,y) \in R_\varepsilon$.

Also note that since $K_\varepsilon \equiv 0$ in Ω, we have $\|K_\varepsilon\|_{H^1(\Omega)\oplus H^1(R_\varepsilon)} = \|K_\varepsilon\|^2_{H^1(R_\varepsilon)}$. Moreover,

$$\begin{aligned}\|K_\varepsilon\|^2_{L^2(R_\varepsilon)} &\leqslant 2\int_0^\varepsilon \int_{\Gamma_\varepsilon^s} |h_\varepsilon(s)|^2 \, |w_\varepsilon(0,y) - v_\varepsilon(0)|^2 \, dsdy \\ &+ 2\int_{1-\varepsilon}^1 \int_{\Gamma_\varepsilon^s} |h_\varepsilon(1-s)|^2 \, |w_\varepsilon(1,y) - v_\varepsilon(1)|^2 \, dsdy \\ &\leqslant C_2 \, \varepsilon^N \, \|w_\varepsilon\|^2_{\mathcal{C}(\overline{\Omega})} \, .\end{aligned}$$

Now note that $h'_\varepsilon(s) = \varepsilon^{-1}h'(x/\varepsilon)$, $h'_\varepsilon(1-s) = -\varepsilon^{-1}h'((1-s)/\varepsilon)$. Hence, with similar estimates as above,

$$\begin{aligned}\|\nabla K_\varepsilon\|^2_{L^2(R_\varepsilon)} &\leqslant 2\int_0^\varepsilon |h'_\varepsilon(s)|^2 \int_{\Gamma_\varepsilon^s} |w_\varepsilon(0,y) - v_\varepsilon(0)|^2 \, dsdy \\ &+ 2\int_{1-\varepsilon}^1 |h'_\varepsilon(1-s)|^2 \int_{\Gamma_\varepsilon^s} |w_\varepsilon(1,y) - v_\varepsilon(1)|^2 \, dsdy \\ &+ 2\int_0^\varepsilon |h_\varepsilon(s)|^2 \int_{\Gamma_\varepsilon^s} |\nabla_y w_\varepsilon(0,y)|^2 \, dsdy + 2\int_{1-\varepsilon}^1 |h_\varepsilon(1-s)|^2 \int_{\Gamma_\varepsilon^s} |\nabla_y w_\varepsilon(1,y)|^2 \, dsdy \\ &\leqslant \mathrm{C} \, \varepsilon^N \, \|w_\varepsilon\|^2_{\mathcal{C}^1(\overline{\Omega})} \, ,\end{aligned}$$

where we have used that $\displaystyle\int_0^\varepsilon \int_{\Gamma_\varepsilon^s} r dsdy = O(\varepsilon^N)$.

The following estimates hold (see [12]), for some $C > 0$,

$$\|w_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \leqslant \frac{C}{|\lambda| + 1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \tag{7.14}$$

$$\|w_\varepsilon\|_{\mathcal{C}^1(\overline{\Omega})} \leqslant \frac{C}{|\lambda|^{1/2} + 1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \, . \tag{7.15}$$

Using (7.15) we have that

$$\|K_\varepsilon\|_{H^1(R_\varepsilon)} \leqslant C \, \frac{\varepsilon^{N/2}}{|\lambda|^{1/2} + 1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \, . \tag{7.16}$$

which shows the first inequality of (7.8).

On the other hand we also have that

$$\left\|\lambda(A_\varepsilon + \lambda)^{-1}K_\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \leqslant |\lambda| \frac{1}{|\lambda|^{1/2} + 1} \, \|K_\varepsilon\|_{L^2(R_\varepsilon)} \leqslant C \, \frac{\varepsilon^{N/2}}{|\lambda|^{1/2} + 1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \tag{7.17}$$

and

$$\begin{aligned}&\|(I - \lambda(A_\varepsilon + \lambda)^{-1})(E_\varepsilon^{\mathcal{C}} - E_\varepsilon)(A_0 + \lambda)^{-1} M \, f_\varepsilon\|_{H^1(\Omega)\oplus H^1(R_\varepsilon)} \\ &\qquad \leqslant \|K_\varepsilon\|_{H^1(\Omega)\oplus H^1(R_\varepsilon)} + \left\|\lambda(A_\varepsilon + \lambda)^{-1}K_\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \\ &\qquad \leqslant C \, \frac{\varepsilon^{N/2}}{|\lambda|^{1/2} + 1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \, .\end{aligned}$$

■

Proof of Lemma 7.4: It follows easily from the definition of the extension E_ε and of the projection M_ε, that $\|E_\varepsilon\|_{\mathcal{L}(L^\infty(\Omega)\oplus L^\infty(0,1),L^\infty(\Omega_\varepsilon))} = 1$ and $\|M_\varepsilon\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega)\oplus L^\infty(0,1))} \leqslant 1$. Hence,

$$\left\|E_\varepsilon A_0(A_0+\lambda)^{-1}M\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega_\varepsilon))} \leqslant C\left\|A_0(A_0+\lambda)^{-1}\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon)\oplus L^\infty(0,1))} \tag{7.18}$$

Let $f=(f_\Omega,f_{R_0})\in\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)$, be such that

$$(A_0+\lambda)^{-1}f=(w,v). \tag{7.19}$$

or equivalently

$$\begin{cases} -\Delta w+\lambda w=f_\Omega, \text{ in } \quad \Omega \\ \dfrac{\partial w}{\partial n}=0, \text{ in } \quad \partial\Omega \\ -\dfrac{1}{g}(gv_s)_s+\lambda v=f_{R_0}, \text{ in } \quad (0,1) \\ v(0)=w(0), \quad v(1)=w(1) \end{cases} \tag{7.20}$$

proceeding as in the proof of Proposition 3.2 (iv), we have that

$$\|w\|_{\mathcal{C}(\overline{\Omega})} \leqslant \frac{C}{|\lambda|+1}\|f_\Omega\|_{\mathcal{C}(\overline{\Omega})}, \quad \|v\|_{\mathcal{C}(\overline{\Omega})} \leqslant \frac{C}{|\lambda|+1}\Big(\|f_\Omega\|_{\mathcal{C}(\overline{\Omega})}+\|f_{R_0}\|_{\mathcal{C}(0,1)}\Big).$$

Since $A_0(A_0+\lambda)^{-1}=I-\lambda(A_0+\lambda)^{-1}$, then

$$\begin{aligned}\left\|A_0(A_0+\lambda)^{-1}f\right\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)} &= \left\|f-\lambda(A_0+\lambda)^{-1}f\right\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)} \\ &\leqslant \|f\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}+C\|f\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)} \\ &\leqslant \widetilde{C}\|f\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}.\end{aligned}$$

Applying this to (7.18), we have that

$$\left\|E_\varepsilon A_0(A_0+\lambda)^{-1}M\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega_\varepsilon))} \leqslant \text{C}, \tag{7.21}$$

where C is independent of λ and ε.

Part (ii) is immediate from the fact that A_ε is positive and self-adjoint. ■

Proof of Lemma 7.5: The proof follows from Lemma 7.2, Lemma 7.3, Lemma 7.4 and statement (3.8) from Proposition 3.3. ■

References

[1] J. M. Arrieta, Spectral Behavior and Uppersemicontinuity of Attractors, *International Conference on Differential Equations,* **1, 2** *(Berlin, 1999), 615–621, World Sci. Publ., River Edge, NJ*, (2000).

[2] J. M. Arrieta and A. N. Carvalho, Spectral convergence and nolinear dynamics of reaction diffusion equations under perturbations of the domain, *J. Differential Equations*, **199** (1) (2004) 143–178.

[3] J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz, Dynamics in Dumbbell Domains I: Continuity of the Set of Equilibria. Continuity of the set of equilibria, *J. Differential Equations*, **231** (2006) 551–597.

[4] J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz, Dynamics in Dumbbell Domains II: The Limiting Problem. Submitted for publication.

[5] J. M. Arrieta, A. Carvalho and A. Rodriguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds, *Comun. Partial differential Equations* **25** (1-2) (2000) 1–37.

[6] S. M. Bruschi, A. N. Carvalho, J. W. Cholewa, T. Dlotko, Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations, *J. Dynam. Differential Equations* **18** (2006), 767-814.

[7] A. N. Carvalho and S. Piskarev A General Approximation Scheme for Attractors of Abstract Parabolic Problems, *Numerical Functional Analysis and Optimization*, **27** (7-8) (2006) 785–829.

[8] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John-Wiley & Sons, New-York (1984).

[9] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25**, American Mathematical Society, Providence 1988.

[10] J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl. (4)* **154**, 281–326, (1989).

[11] D. B. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer Verlag, New York, 1981.

[12] X. Mora, *Semilinear parabolic problems define semiflows on C^k spaces*, Trans. Amer. Math. Soc. **278** (1) (1983) 21–55.

[13] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag (1983).

[14] G. Vainikko, Approximative methods for nonlinear equations (two approaches to the convergence problem) *Nonlinear Anal.* **2** (6) (1978) 647–687.